EE 131A Midterm Solution

Probability

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1. **Problem 1. (total: 25pts)** In a game of poker five cards are picked at random from a deck of 52 cards. Note that a deck of cards has 13 denominations (or kinds) (namely, Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, and King) and 4 suits (namely, Hearts, Spades, Clubs, and Diamonds). What is the probability of being dealt

   (a) (5pts) A flush? That is, all five cards are from the same suit. So, Ace, 10, Jack, Queen and King of Spades will comprise a flush.

   (b) (5pts) One pair? This occurs when the cards have denominations a, a, b, c, d, where a, b, c, and d, are all distinct. So 10 of Spades, 10 of Hearts, 2 of Clubs, 3 of Diamonds, and Jack of Diamonds will comprise One Pair.

   (c) (5pts) Two Pairs? This occurs when the cards have denominations a, a, b, b, c, where a, b, and c, are all distinct. So 10 of Spades, 10 of Hearts, 2 of Clubs, 2 of Hearts, and Jack of Diamonds will comprise Two Pair.

   (d) (5pts) Three of a kind? This occurs when the cards have denominations a, a, a, b, c, where a, b, and c, are all distinct. So 10 of Spades, 10 of Hearts, 10 of Clubs, 2 of clubs, and Jack of Diamonds will comprise Three of a kind.

   (e) (5pts) Four of a kind? This occurs when the cards have denominations a, a, a, a, b, where a and b, are distinct. So 10 of Spades, 10 of Hearts, 10 of Clubs, 10 of Diamonds, and Jack will comprise Four of a kind.

   Solution:

   (a) The total number of possible hands is $\binom{52}{5}$. To calculate the number of hands where we ca have a flush, we need to pick a suit and choose possible cards from that suit:

$$\frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}}$$

   (b) To calculate the number of possible hands with one pair in it we first choose the denomination of the pair and then three other denominations, and the suits of the pair, and the suits of the three other cards:

$$\frac{13\binom{12}{3}\binom{4}{2}4^3}{\binom{52}{5}}$$

(c) Now we need to choose the denominations of a,b and then choose c and the suits for pairs and the other card:

$$\frac{\binom{13}{2}\binom{11}{1}\binom{4}{2}^2\binom{4}{1}}{\binom{52}{5}}$$

(d) We first choose the denomination of a and then b,c and the suits of the three "a"s and the other cards:

$$\frac{13\binom{12}{2}\binom{4}{3}4^2}{\binom{52}{5}}$$

(e) We need to choose a and then b and the suit of the other card (There is only one possible set for the suits of the 4 "a"s as we only have 4 suits):

$$\frac{13 \times 12 \times 4}{\binom{52}{5}}$$

2. **Problem 2 (total: 20 pts)** A woman has n distinct keys, of which only one will open her door.

(a) (10 pts) If she tries the keys at random, each time discarding those that do not work, what is the probability that she will open the door on her k^{th} try?

(b) (10 pts) If she does not discard the previously tried keys (that is, each time she tries a randomly picked key from the entire set of n keys.) then what is the probability that she will open the door on her k^{th} try?

Solution:

(a) Every time we try a key, if it doesn't work we put it away and try another one, so in the k^{th} try there are $n-k+1$ keys left and the probability of success is $\frac{1}{n-k+1}$ and the probability of failure is $\frac{n-k}{n-k+1}$. If N is the number of tries needed, $P(N=k)$ will be equal to the probability of failure in the first $k-1$ trials and success at the k^{th} try, which can be calculated as:

$$P(N=k) = \frac{n-1}{n}.\frac{n-2}{n-1}.\frac{n-3}{n-2}...\frac{n-k+2}{n-k+3}.\frac{n-k+1}{n-k+2}.\frac{1}{n-k+1} = \frac{1}{n}$$

This is similar to the notion that in a random ordering of the keys, the correct key could be placed anywhere from first to n_{th} position with equal probability $1/n$

(b) Now every time we try a key we throw it back in the pool, so the probability of success in each try is $\frac{1}{n}$ and the probability of failure in each try is $\frac{n-1}{n}$. So now

$P(N=k)$ is the probability of failure in $k-1$ trials and success in the k^th try calculated as:

$$P(N=k) = \left(\frac{n-1}{n}\right)^{k-1}\frac{1}{n}$$

3. **Problem 3. (20pts)** A worker has asked her supervisor for a letter of recommendation for a new job. She estimates that (i) with probability 0.8 she will get the job if she receives a Strong recommendation, (ii) with probability 0.4 she will get the job if she receives a Moderate recommendation, and (iii) with probability 0.1 she will get the job if she received a Weak recommendation. She further estimates that the probabilities the recommendation will be Strong, Moderate, or Weak are 0.7, 0.2 and 0.1, respectively.

   a. (10 pts) What is the probability that she will receive a the new job offer?

   b. (5 pts) Given that she does receive the offer, what is the probablility that she received a Strong recommendation?

   c. (5 pts) Given that she does not receive the offer, what is the probability that she received a Weak recommendation?

   Solution:

   subitem (a) We can name O, R_s, R_m, R_w as the events of getting a job offer, and having a Strong, Moderate, or Weak recommendation. We have:

   $P(O) = P(O|R_s)P(R_s) + P(O|R_m)P(R_m) + P(O|R_w)P(R_w)$
   $= 0.8 \times 0.7 + 0.4 \times 0.2 + 0.1 \times 0.1 = 0.65$

   (b) Using the Bayes rule we can write:

   $P(R_s|O) = \dfrac{P(O|R_s)P(R_s)}{P(O)}$
   $= \dfrac{0.8 \times 0.7}{0.65} = 0.8615$

   (c)
   $P(R_w|O^c) = \dfrac{P(O^c|R_w)P(R_w)}{P(O^c)}$
   $\dfrac{(1-0.1) \times 0.1}{1-0.65} = 0.2571$

4. **Problem 4.(total: 20 pts)** A gambling book recommends the following winning strategy for the game of roulette. It recommends that a gambler bet \$1 on Red. If Red appears (which has probability of 18/38) then the gambler should take her \$1 profit and quit. If the gambler loses this bet (which has probability 20/38 of occurring), she should make additional \$1 bets on red on each of the next two spins of the roulette

wheel and the quit. Let X be the random variable that denotes the gamblers winnings when she quits.

a) (5 pts) State all the outcomes in the underlying sample space in terms of wins and losses. For example, if she loses her first bet, but wins the two next ones, then the outcome is LWW.

b) (10 pts) What are the values that the random variable X can take and what are their respective probabilities? What is $P[X > 0]$?

c) (5 pts) What is $E[X]$? Is this a winning strategy?

Note 1: To answer this question, you do not need to know anything more about the game of roulette than the information provided here.

Note 2: When the player bets \$1 on Red, if Red appears then she makes a profit of \$1, i.e., her winning from this bet is +\$1. If Red does not appear, then she loses the \$1 she bet, i.e., her winning from this bet is -\$1.

Solution:

a) All outcomes are:

$$\{W, LLL, LLW, LWL, LWW\}$$

b) From Note 2, we know that we will have a $+1$ profit for each win, and -1 for each lose. Applying to preceded result, We have,

$$X \in \{-3, -1, 1\}$$

, And,

$$P[X = -3] = P[\text{LLL}] = (20/38)^3 \approx 0.1458$$

$$P[X = -1] = P[\text{LLW or LWL}] = 2 \times (18/38)(20/38)^2 \approx 0.2624$$

$$P[X = 1] = P[\text{W or LWW}] = (18/38) + (18/38)^2(20/38) \approx 0.5918$$

Now, we have,

$$P[X > 0] = P[X = 1] \approx 0.5918$$

c)

$$\begin{aligned} E[X] &= -3 \times P[X = -3] - 1 \times P[X = -1] + 1 \times P[X = 1] \\ &\approx -0.108 \end{aligned}$$

5. **Problem 5. (total: 15 pts)** An LCD display has 1000×750 pixels. A display is accepted if it has 15 or fewer faulty pixels. The probability that a pixel is faulty coming out of the production line is 10^{-5}. Using the Poisson approximation find the proportion of displays that are accepted.

Solution: In each display, there are 1000×750 pixels, and the faulty prob for each pixel is 10^{-5}, so that, we have,

$$\alpha = 1000 \cdot 750 \cdot 10^{-5} = 7.5$$

Using Poisson's pmf, we get the proportion of accepted displays P,

$$P = \sum_{n=0}^{15} \frac{7.5^n}{n!} e^{-7.5} \approx 0.9954$$