Probability
Instructor: Professor Roychowdhury

1. Problem 1. (total: 20pts) From a group of 5 women and 7 men, a committee is to be formed by picking 5 members randomly from this group of 12 individuals:

   (a) (5pts) What is the probability that the committee has exactly 2 women and 3 men?

   (b) (10pts) Two men, Mr. Jones and Mr. Chen, are sworn enemies. What is the probability that the committee has two women and 3 men, but not both Mr Jones and Mr. Chen?

   (c) (5pts) Given that the randomly picked committee has 2 women and 3 men, what is the conditional probability that the committee does not have both Mr Jones and Mr. Chen?

   Solution:

   (a) Let A be the event that the committee has exactly 2 women and 3 men, we have,

$$P[A] = \frac{\binom{5}{2}\binom{7}{3}}{\binom{12}{5}} = \frac{175}{396}$$

   (b) Let B be the event that the committee has two women and 3 men, but not both Mr Jones and Mr. Chen. We have,

$$P[B] = \frac{\binom{5}{2}(\binom{5}{3} + \binom{2}{1}\binom{5}{2})}{\binom{12}{5}} = \frac{25}{66}$$

   (c)

$$P[B|A] = \frac{P[B \cap A]}{P[A]} = \frac{P[B]}{P[A]} = \frac{6}{7}$$

2. Problem 2 (total: 25 pts) A bin contains three different types of disposable flashlisghts. The probability that a type 1 flashlisght will give over 100 hours of use is 0.7; the corresponding probabilities for type 2 and and 3 flashlisghts being 0.4 and 0.3, respectively. Suppose that 20% of the flashlights in the bin are type 1, 30% are type 2, and 50% are type 3.

   (a) (10pts) What is the probability that a randomly chosen flashlight will give more than 100 hours of use?

   (b) (15 pts) Given that a randomly picked flashlight lasts over 100 hours, what is the conditional probability that that it is of type j flashlight for j= 1, 2, and 3?

   Solution:

(a) Let A_k be the event that the flashlight is of type k, and B be the event that the flashlight will give more than 100 hrs of use, we have,

$$\begin{aligned} P[B] &= P[B|A_1]P[A_1] + P[B|A_2]P[A_2] + P[B|A_3]P[A_3] \\ &= 0.2 \cdot 0.7 + 0.3 \cdot 0.4 + 0.5 \cdot 0.3 = 0.41 \end{aligned}$$

(b)

$$\begin{aligned} P[A_1|B] &= \frac{P[B|A_1]P[A_1]}{P[B]} \\ &= \frac{14}{41} \end{aligned}$$

(b)

$$\begin{aligned} P[A_2|B] &= \frac{P[B|A_2]P[A_2]}{P[B]} \\ &= \frac{12}{41} \end{aligned}$$

(b)

$$\begin{aligned} P[A_3|B] &= \frac{P[B|A_3]P[A_3]}{P[B]} \\ &= \frac{15}{41} \end{aligned}$$

3. Problem 3. (15pts) A laboratory blood test is 95% effective in detecting a certain disease when it is in fact present. (That is, if a person with the disease is tested then with probability 0.95 the test will come out positive). However, the test also yields a false positive result for 1% of the healthy persons tested. (That is, if a healthy person is tested, then with probability 0.01, the test will imply he or she has the disease, i.e., the test will be positive.)

   If 0.5% of the population actually has the disease, what is the probability a randomly tested person actually has the disease given that the test result is positive.
   Solution: Let A = {The tested person has disease}.
   T = {Test result is positive},

   We have,

$$\begin{aligned} P[A|T] &= \frac{P[T|A]P[A]}{P[T|A]P[A] + P[T|A^c]P[A^c]} \\ &= \frac{0.95 \cdot 0.005}{0.95 \cdot 0.005 + 0.01 \cdot 0.995} \\ &= \frac{95}{294} \end{aligned}$$

4. Problem 4. (total: 30 pts) The game of craps is played as follows: A player rolls two fair dice. If the sum of the dice is either 2, 3, or 12, the player loses; if the sum is either a 7 or an 11, he or she wins. If the outcome is anything else, the player continues to roll the two dice until he or she rolls either the initial outcome or a 7. If the 7 comes first, the player loses; whereas if the initial outcome reoccurrs before the 7, the player wins.

   (a) (5 pts) What is the probability of winning on the first roll of the two dice?

   (b) (5 pts) What is the probability of losing on the first roll of the two dice?

   (c) (8 pts) Suppose that the first roll of the two dice yields a sum of 5. What is the probability that the player wins? [Hint: use a geometric law]

   (d) (7 pts) Suppose that the first roll of the two dice yields a sum of 6. What is the probability that the player wins? [Hint: use a geometric law]

   (e) (5 pts) Provide a strategy (not necessarily a complete solution) for computing the winning probability of a player at the game of craps.

   Solution: let us denote X_i be to the sum of rolling two fair dice. For example: $P[X_i = 5] = P[\{1,4\},\{2,3\},\{3,2\},\{4,1\}] = \frac{4}{36}$, then full table of its pmf could be found as follows:

X_i=k	2	3	4	5	6	7	8	9	10	11	12
$P[X_i = k]$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{35}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

   (a) The probability of wining on the first roll of the two dice is then given by:

$$P[\text{winning on the first roll}] = P[X_1 = 7] + P[X_1 = 11] = \frac{2}{9}$$

   (b) The probability of losing on the first roll of the two dice is then given by:

$$P[\text{losing on the first roll}] = P[X_1 = 2] + P[X_1 = 3] + P[X_1 = 12] = \frac{1}{9}$$

   (c) Given $X_1 = 5$, then according to the pmf of X_i , we can write the probability of winning in the nth round:

$$P[\text{wins at nth round}] = (\frac{4}{36})(1 - \frac{4}{36} - \frac{6}{36})^{n-1}$$

then the total probability of winning is given by:

$$P[\text{winning}] = \sum_{k=1}^{\infty}(\frac{26}{36})^k \frac{4}{36} = \frac{1}{1-\frac{26}{36}} \frac{4}{36} = \frac{2}{5}$$

Also we could solve it recursively by:

$$P[winning] = \frac{4}{36} + \frac{26}{36} P[winning]$$

(d) Given $X_1 = 6$, then according to the pmf of X_i , we can write the probability of winning in the nth round:

$$P[\text{wins at nth round}] = (\frac{5}{36})(1 - \frac{5}{36} - \frac{6}{36})^{n-1}$$

then the total probability of winning is given by:

$$P[\text{winning}] = \sum_{k=1}^{\infty} (\frac{25}{36})^k \frac{5}{36} = \frac{1}{1 - \frac{25}{36}} \frac{5}{36} = \frac{5}{11}$$

Also we could solve it recursively by:

$$P[winning] = \frac{5}{36} + \frac{25}{36} P[winning]$$

(e) Similarly we can compute the probability of winning as follows:

$$P[winning|X_1 = 4] = \frac{1}{3}$$

$$P[winning|X_1 = 8] = \frac{5}{11}$$

$$P[winning|X_1 = 9] = \frac{2}{5}$$

$$P[winning|X_1 = 10] = \frac{1}{3}$$

therefore the wining probability of a player at the game of craps is given by:

$$\begin{aligned} P[\text{winning}] &= \sum_{k=2}^{12} P[\text{winning}|X_1 = k] P[X_1 = k] \\ &= \frac{8}{36} + (\frac{1}{3} \times \frac{3}{36} + \frac{5}{11} \times \frac{5}{36} + \frac{2}{5} \times \frac{4}{36}) \times 2 \\ &= \frac{244}{495} \\ &= 0.4929 \end{aligned}$$

5. Problem 5. (total: 15 pts) In a randomly shuffled full deck (i.e., comprising all 52 cards), a player flips through the deck until the Ace of spades shows up.

   (a) (8 pts) What is the probability that the very next card is a Two of clubs?

   (b) (7 pts) What is the probability that the very next card is an Ace of clubs?

   Solution:

   (a) the total number of possible shuffling between 52 cards is 52!. Consider Ace of Spades and Two of Clubs as a single card, then we have 51 cards to shuffle, the total number of combination is 51!, therefore the probability that the very next card is a Two of Clubs is given by:

$$p = \frac{51!}{52!} = \frac{1}{52}$$

(b) same as part(a), the total number of possible shuffling between 52 cards is 52!. Consider Ace of Spades and Ace of Clubs as a single card, then we have 51 cards to shuffle, the total number of combination is 51!, therefore the probability that the very next card is an Ace of Clubs is given by:

$$p = \frac{51!}{52!} = \frac{1}{52}$$