EE 131A — Midterm Solutions
Instructor: Professor Roychowdhury — November 2016

1. A computer reserves a spot in a network for 10 minutes. To extend the reservation the computer must successfully send a refresh message. However, messages are lost indpendently with probability $\frac{1}{4}$. Suppose that it takes 10 seconds to send a refresh request and receive an acknowledgment.

   (a) What is the expected number of refresh requests?

   (b) Assuming that the spot is reserved at time t=0, what is the latest time the computer should start sending refresh messages in order to have at least a 99% chance of successfully extending the reservation time beyond the allotted ten minutes?

   Solution:

   (a) Let X be the number of refresh requests. The probability that the k^{th} refresh attempt is successful is

   $$P(X=k) = p(1-p)^{k-1},$$

   where $p = \frac{3}{4}$ is the probability of a successful request. We know that the expected value of a geometric random variable is $\frac{1}{p}$, which in this case is $\frac{4}{3}$

   (b) We need to determine k such that

   $$p(X \leq k) \geq 0.99,$$

   and we know for a geometric random variable that $p(X > k) = (1-p)^k$. With $k = 4$, we can guarantee that $p(X > k) < 0.01$ is satisfied. Thus we need to start sending refresh requests 40 seconds before the 10 minutes allotted time is finished.

2. (a) Two randomly drawn cards are removed from an ordinary deck of 52 cards.

      i. What is the probability that a randomly drawn card from this reduced deck is the 2 of spades?

      ii. What is the probability that a randomly drawn card from this reduced deck is a spade?

   (b) The two cards are put back in, and the full deck of cards is shuffled to create a random order. What is the probability that the kth card is the first Ace, and the (k+1)th card is the 2 of clubs?

   Solution:

   (a) In this part two random cards are removed from the deck.

      i. Let A be the event that the randomly drawn card is 2 of spades. Also, let B be the event that 2 of spades was removed from the deck. Thus,

      $$p(A) = P(A|B)p(B) + P(A|B^c)p(B^c) = P(A|B^c)p(B^c) = \frac{1}{50} \times \frac{\binom{51}{2}}{\binom{52}{2}} = \frac{1}{52}$$

ii. Let A be the event that the randomly drawn card is a spades. Let B_i be the event that there are i spade cards among the two cards removed from the deck, for $0 \le i \le 2$. Thus,

$$p(A) = \sum_{i=0}^{2} P(A|B_i)p(B_i) = \frac{13}{50} \times \frac{\binom{13}{0}\binom{39}{2}}{\binom{52}{2}} + \frac{12}{50} \times \frac{\binom{13}{1}\binom{39}{1}}{\binom{52}{2}} + \frac{11}{50} \times \frac{\binom{13}{2}\binom{39}{0}}{\binom{52}{2}} = \frac{1}{4}$$

(b) We can treat this problem as that of ordering 52 cards in such a way that the constraint of the problem is satisfied. If there is no constraint what so ever, there are 52! possibilities to arrange the cards in order. For arranging the cards according to the constraint, we first take out all the aces and the 2 of clubs. Then we choose one of the aces to be the first ace and be placed in the k^{th} position, which can be done in $\binom{4}{1}$ ways. We then put the first ace and the 2 of clubs in the k^{th} and $k+1^{\text{th}}$ positions, respectively. Then we determine where to put the rest of the aces after the first ace, which can be done in $\binom{52-(k+1)}{3}$ ways. Then we can place rest of the cards in arbitrary order in the remaining $52-5$ spots in 47! ways. Thus, the desired probability is

$$\frac{\binom{4}{1}\binom{51-k}{3}47!}{52!}$$

3. In answering a question on a multiple-choice test, a student either knows the answer or guesses. When answering different questions, she knows the answers independently each time with probability p, and guesses with probability $1-p$. Assume that a student who guesses at the answer will be correct with probability $\frac{1}{m}$, where m is the number of multiple-choice alternatives.

(a) What is the probability of answering a particular problem correctly?

(b) In an exam with n problems, what is the expected number of problems the student would get correctly?

(c) What is the conditional probability that a student knew the answer to a particular question given that she answered it correctly?

(d) Now suppose a student answers k questions out of n correctly. What is the probability that she knew answers to all k of them?

Solution:

(a) Let A be the event the the student answers the problem correctly and let B be the event that she knows the answer.

$$P(A) = P(A|B)p(B) + P(A|B^c)p(B^c) = 1 \times p + \frac{1}{m} \times (1-p) = \frac{1+(m-1)p}{m} \triangleq q$$

(b) The student answers each problem correctly with the same probability as computed above independently of the other problems. Since there are n problems in total, the number of problems he answers correctly(N) is a binomial random variable, $N \sim \text{Bin}(n, q)$, whose expected value is $nq = \frac{n(1+(m-1)p)}{m}$.

(c) Again, let A be the event the the student answers the problem correctly and let B be the event that she knows the answer. We want to compute $P(B|A)$ and we use Bayes law.

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)} = \frac{1 \times p}{q} = \frac{mp}{1 + (m-1)p}$$

(d) $\{N = k\}$ is the event that the student answers k questions out of n correctly and let C be the event that she knows the answer to all k questions. Thus,

$$P(C|N = k) = \frac{P(N = k|C)P(C)}{P(N = k)}$$

We know that $P(N = k) = \binom{n}{k} q^k (1-q)^{n-k}$. The probability that she knows the answer to all the k specific questions is $p(C) = p^k$, because he knows the answer to each problem with probability p independent of the others. Finally $P(N = k|C)$ is equal to the probability that he answers the rest of the questions incorrectly which is $(1-q)^{n-k}$. Therefore,

$$P(C|N = k) = \frac{(1-q)^{n-k} p^k}{\binom{n}{k} q^k (1-q)^{n-k}} = \frac{1}{\binom{n}{k}} (\frac{p}{q})^k$$

4. Suppose at the production line of a company, every package contains n items. Suppose a package is known to contain one defective item. Therefore, items in that product package are to be tested until the defective item is discovered. If the tester pulls out the items at random without replacement,

   (a) What is the probability that she will test the defective item in the k^{th} draw?

   (b) What is the probability that a specific non-defective item is not tested?

   Solution:

   (a) This is equivalent to asking in an arrangement of n items in a line, what is the probability that a specific item is place in the k^{th} position, which is trivially $\frac{1}{n}$ regardless of k.

   (b) This is in fact equivalent to asking in an arrangement of n items in a line, what is the probability that a specific item, A, is placed after another specific item, B. By symmetry, this probability is equal to $\frac{1}{2}$, because in a random arrangement of n objects, half of the times A is placed after B and half of the times A is placed after B.

   One can prove this using part (a) too. Let A be the event of interest. Suppose that you keep drawing items until all the items are out. And suppose that the defective item appears in the k^{th} draw, $1 \leq k \leq n-1$. Then you can determine where to have the specific operative item in the remaining $n-k$ draws in $\binom{n-k}{1}$ ways and the remaining $n-2$ items could appear in arbitrary order in the other draws in $(n-2)!$ ways. Thus,

$$p(A) = \sum_{k=1}^{n-1} \frac{(n-k)(n-2)!}{n!} = \frac{1}{n(n-1)} \sum_{k=1}^{n-1} k = \frac{1}{2}$$

5. A page of text has 100 words and there is a typo in each word with probability 0.01. Using a Poisson approximation for the number of typos in the text,

(a) What is the probability that the number of typos is one standard deviation more than the average number of typos?

(b) What is the probability that an exam sheet has 3 typos given that it has at least one typo?

Solution: Let X be the number of typos in the page. Then using the Poison approximation we assume that $X \sim Poi(\lambda)$, where $\lambda = 100 \times 0.01 = 1$

(a) We know that for a Poison random variable, the expected value and the standard deviation are equal to λ and $\sqrt{\lambda}$, respectively. Thus,

$$P(X > 1+1) = 1 - P(X \leq 2) = 1 - \sum_{i=0}^{2} \frac{e^{-1}}{i!} = 1 - \frac{5e^{-1}}{2}$$

(b)

$$P(X = 3 | X \geq 1) = \frac{p(X = 3, X \geq 1)}{p(X \geq 1)} = \frac{p(X = 3)}{1 - p(X = 0)} = \frac{\frac{e^{-1}}{3!}}{1 - e^{-1}}$$

6. A hacker is trying to break down a 4-character password that only consists of $0-9$ digits.

   (a) If she has the side information that sum of the digits is 10, how many possibilities are there for him to check?

   (b) In order to help him generate these sequences in an automated manner, he does the following: He creates an array with 13 entries, and randomly picks 3 entries. This divides the array into four parts, and the entry counts in the respective parts are the digits to try. Thus, if the picked locations are: 3, 4, 7, then the first part has two entries, and hence the first digit is 2; the second part is empty, and so the second digit is 0; the third part has two entries and thus the third digit is 2, and the last part has 6 entris, and so the last digit is 6. So she will try (2,0,2,6).

      i. What is the probability that she tries a sequence where the first digit is 0?

      ii. What is the probability that she tries a sequence where the fist two digits are 0's?

      iii. What is the probability that she tries a sequence where the second digit is 0?

Solution:

(a) We take the approach described in part (b)! Let x_1, x_2, x_3 and x_4 represent the first, second, third and the forth digits of the number(leave zero for the missing digits if not a 4-digits number). Then the number of such integers for which sum of the digits is 10, is the number of non-negative solutions to the equation $x_1 + x_2 + x_3 + x_4 = 10$, which is $\binom{10+3}{3}$, except those solutions in which one of the variables is equal to 10 and the rest are 0. There are 4 such solutions. So in total, we have

$$\binom{13}{3} - 4 = 282$$

possibilities.

(b) Using the same approach, we note that having a specific set to 0 corresponds to having its respective variable x_i to be set to zero.

i. This is equivalent to having $x_1 = 0$. Thus we are interested in the number of non-negative solutions to the equation $x_2 + x_3 + x_4 = 10$, where all the variables are less that 10. With the same argument as in part (a), we have

$$\binom{12}{2} - 3 = 63$$

possibilities in which $x_1 = 0$. Since all these possibilities are equally likely, the desired probability is $\frac{63}{282}$.

ii. This is equivalent to $x_1 = 0$ and $x_2 = 0$. Thus we are interested in the number of non-negative solutions to the equation $x_3 + x_4 = 10$, where all the variables are less that 10. With the same argument as in part (a), we have

$$\binom{11}{1} - 2 = 9$$

Again, since all these possibilities are equally likely, the desired probability is $\frac{9}{282}$.

iii. By symmetry, it is the same as in part (i), so, there are

$$\binom{12}{2} - 3 = 63$$

possibilities and the desired probability is $\frac{63}{282}$.