1. DNA fingerprint (Hint: Poisson approximation of Binomial distribution)
In a certain style of detective fiction, the sleuth is required to declare the criminal has the unusual characteristics; find this person and you have your man. Assume that any given individual has these unusual characteristics with probability 10^{-7} independently of all other individuals, and that the city in question contains 10^{7} inhabitants.
a) Given that the police inspector finds such a person, what is the probability that there is at least one other?
b) If the inspector finds two such people, what is the probability that there is at least one more?
c) How many such people need to be found before the inspector can be reasonably confident that he has found them all?

Binomial w/ $p = 10^{-7}$, $n = 10^{7}$ let k = # such person

a) $$P(k \geq 2 \mid k \geq 1) = \frac{P(k \geq 2)}{P(k \geq 1)} = \frac{1 - P(k=1) - P(k=0)}{1 - P(k=0)} = \frac{1 - \binom{n}{1}(1-p)^{n-1}p - (1-p)^n}{1-(1-p)^n}$$

≈ 0.41802328097

b) $$P(k \geq 3 \mid k \geq 2) = \frac{1 - \binom{n}{2}(1-p)^{n-2}p^2 - \binom{n}{1}(1-p)^{n-1}p - (1-p)^n}{1 - \binom{n}{1}(1-p)^{n-1}p - (1-p)^n}$$

(c) $P(k > ?) \approx 0$

Poisson Approximation: $\alpha = np = 1$ (Why?)

$P(k=0) = \frac{1^0 e^{-1}}{0!} = \frac{1}{e}$ $P(k=1) = \frac{1^1 e^{-1}}{1!} = \frac{1}{e}$ $P(k=2) = \frac{1^2 e^{-1}}{2!} = \frac{1}{2e}$

$P(k=3) = \frac{1^3 e^{-1}}{3!} = \frac{1}{6e}$ $P(k=4) = \frac{1}{4!}\frac{1}{e} = \frac{1}{24e}$ $P(k=5) = \frac{1}{5!}\frac{1}{e} = \frac{1}{120e}$

...

x	$P(k > x)$
0	0.632121
1	0.264241
2	0.0803014
3	0.0189882
4	0.00365985
5	0.000594185
⋮	

2. A high school student is anxiously waiting to receive mailing telling her whether she has been accepted to a certain college. She estimates that the conditional probability, given that she is accepted and that she is rejected, of receiving a notification on each day of the next week is as follows:

Day	P(mail | accepted)	P(mail | rejected)
Monday	.15	.05
Tuesday	.20	.10
Wednesday	.25	.10
Thursday	.15	.15
Friday	.10	.20

She estimates that her probability of being accepted is 0.6

a. What is the probability that mail is received on Monday?

b. What is the conditional probability that mail is received on Tuesday given that it is not received on Monday?

c. If there is no mail through Wednesday, what is the conditional probability that she will be accepted?

d. What is the conditional probability that she will be accepted if mail comes on Thursday?

e. What is the conditional probability that she will be accepted if no mail arrives that week?

a. $P(\text{mail on Monday}) = 0.6 \times 0.15 + (1-0.6) \times 0.05 = 0.11$

b. $P(\text{mail on Tuesday}) = 0.6 \times 0.2 + (1-0.6) \times 0.1 = 0.16$

$$P(\text{mail on } T \mid \text{no mail on } M) = \frac{0.16}{1-0.11}$$

c. $$P(\text{accepted} \mid \text{no mail through } W) = \frac{0.6(1-0.15-0.2-0.25)}{1-0.11-0.16-0.6\times0.25-(1-0.6)\times0.1} =$$

d. $$P(\text{accepted} \mid \text{mail on } R) = \frac{0.6\times0.15}{0.6\times0.15+0.4\times0.15} = 0.6$$

e. $$P(\text{accepted} \mid \text{no mail}) = \frac{0.6(1-0.15-0.2-0.25-0.15-0.1) = 0.09}{0.09+0.4(1-0.05-0.1-0.1-0.15-0.2)} = \frac{0.09}{0.09+0.16} = \frac{9}{25}$$

3. A page of text has 100 words and there is a typo in each word with probability 0.01. Using a Poisson approximation for the number of typos in the text,
a) What is the probability that the number of typos is more than two?
b) What is the probability that the page has 3 typos given that it has at least one typo?

Poisson Distribution w/ parameter $100 \times 0.01 = 1$

a) $P(k>2) = 1 - P(k=0) - P(k=1) - P(k=2) = 1 - \sum_{i=0}^{2} \frac{1^i e^{-1}}{i!}$

b) $P(k=3 \mid k \geq 1) = \frac{P(k=3, k\geq 1) = P(k=3)}{P(k \geq 1)} = \frac{P(k=3)}{1 - P(k=0)} = \frac{1^3 e^{-1}/3!}{1 - 1^0 e^{-1}/0!} = \frac{1}{6} \frac{1}{e-1}$

4. Show that if three events A, B and C are independent, then A and $(B \cup C)$ are independent

Given : $P(A \cap B) = P(A)P(B)$ $\quad P(A \cap C) = P(A)P(C)$ $\quad P(B \cap C) = P(B)P(C)$

and $\quad P(A \cap B \cap C) = P(A)P(B)P(C)$

Then

$$P[A \cap (B \cup C)] = P[(A \cap B) \cup (A \cap C)] = P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C))$$
$$= P(A)P(B) + P(A)P(C) - P(A \cap B \cap C) = P(A)[P(B) + P(C) - P(B)P(C)]$$
$$= P(A)P(B \cup C) \quad \square$$

5. A football team consists of 20 offensive and 20 defensive players. The players are to be paired in groups of 2 for the purpose of determining roommates.

1) If the pairing is done at random, what is the probability that there are no offensive-defensive roommate pairs?

   a. (Basic) How many ways of pairing are there in total?

$$\frac{(20+20)!}{2^{20}\ 20!}$$

   b. (Medium) How many ways among them satisfies that there are no offensive-defensive roommate pairs?

$$\left(\frac{\binom{20}{10} 10!}{2^{10}} \right)^2$$

2) (Harder) What is the probability that there are 2i offensive-defensive roommate pairs, i=1,2,..10?

$$\binom{20}{2i} \binom{20}{2i} (2i)! \left(\frac{\binom{20-2i}{10-i} (10-i)!}{2^{10-i}} \right)^2 \Big/ \frac{(20+20)!}{2^{20}\ 20!}$$

6. A manufacturer sells transistors wherewith the probability of $p = 0.01$ each one is independently broken. This company sells packages containing 10 transistors and guarantees to return if there are more than one broken transistors per package. How many percentage of sold packages are returned?

Binomial:

$$P(\text{more than 1 broken}) = 1 - P(k=0) - P(k=1)$$
$$= 1 - \begin{bmatrix}10\\0\end{bmatrix}(1-0.01)^{10} - \begin{bmatrix}10\\1\end{bmatrix} 0.01 \cdot (1-0.01)^{9}$$

7. What's the probability of $P(A \cap B|A)$ and $P(A|A \cap B)$

- $P(A \cap B | A) = \dfrac{P(A \cap B \cap A)}{P(A)} = \dfrac{P(A \cap B)}{P(A)} = P(B|A)$

- $P(A | A \cap B) = \dfrac{P(A \cap A \cap B)}{P(A \cap B)} = 1$