UCLA Electrical Engineering Department

ECE131A: Probability and Statistics

MIDTERM EXAMINATION

Winter 2018

Tuesday, February 13, 2018

Exam Duration: 1 hr. 30 min.
Total: 50 points

NAME: ________SOLUTION____

UID: ____________________________________

	1	2	3	4	Total
Marks Obtained	15	9	11	15	50
Maximum Marks	**15**	**9**	**11**	**15**	**50**

I understand that academic integrity is highly valued at UCLA. Further, I understand that academic dishonesty such as cheating and plagiarism, are violations of University Policy and will be pursued by the appropriate campus administrator. Finally, my signature below signifies that the work included is my own, and I completed this assignment honestly.

Signature: ____

Instructions:

(i) Books, cheat sheet, formulae sheet, Calculators, Cell Phones, Computer, Laptops, Tablets, Programmable watches, and IPods are **NOT** allowed

(ii) Provide your solutions only within the space provided within this booklet. Your answers must be legible and easy to follow.

(iii) PLEASE MAKE SURE THAT YOU HAVE 5 PAGES OF THIS EXAMINATION BOOKLET.

FORMULAE SHEET

Commutative properties:

$$A \cup B = B \cup A \quad \text{and} \quad A \cap B = B \cap A. \tag{2.1}$$

Associative properties:

$$A \cup (B \cup C) = (A \cup B) \cup C \quad \text{and} \quad A \cap (B \cap C) = (A \cap B) \cap C. \tag{2.2}$$

Distributive properties:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C). \tag{2.3}$$

By applying the above properties we can derive new identities. DeMorgan's rules provide an important such example:

DeMorgan's rules:

$$(A \cup B)^c = A^c \cap B^c \quad \text{and} \quad (A \cap B)^c = A^c \cup B^c \tag{2.4}$$

$$P\left[\bigcup_{k=1}^{n} A_k\right] = \sum_{j=1}^{n} P[A_j] - \sum_{j<k} P[A_j \cap A_k] + \cdots + (-1)^{n+1} P[A_1 \cap \cdots \cap A_n].$$

Axiom I $\quad 0 \le P[A]$

Axiom II $\quad P[S] = 1$

Axiom III $\quad$ If $A \cap B = \emptyset$, then $P[A \cup B] = P[A] + P[B]$.

Axiom III' $\quad$ If $A_1, A_2, \ldots$ is a sequence of events such that $A_i \cap A_j = \emptyset$ for all $i \neq j$, then

$$P\left[\bigcup_{k=1}^{\infty} A_k\right] = \sum_{k=1}^{\infty} P[A_k].$$

Events	Sets	Notations
A and B cannot occur simultaneously	A and B are disjoint	$A \cap B = \emptyset$
No two of the events $A_1, A_2, \ldots$ can occur simultaneously	A_i, i=1,2,… are pair-wise disjoint	$A_i \cap A_j = \emptyset$, $i \neq j$
A implies B	B contains A or A is contained in B	$A \subset B$

$$P(B) = P\left(\bigcup_{i=1}^{n}(B \cap A_i)\right) = \sum_{i=1}^{n} P(B \cap A_i) = P(B \cap A_1) + \cdots + P(B \cap A_n).$$

$$P\left[\bigcup_{k=1}^{n} A_k\right] \le \sum_{k=1}^{n} P[A_k] \longleftarrow \text{Union bound}$$

Equivalent form of independence

$$P(A) = P(A|B) \Leftrightarrow P(A)P(B) = P(A \cap B) \Leftrightarrow P(B) = P(B|A)$$

Bayes Rule:

$$P(A_j|B) = \frac{P(B \cap A_j)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\sum_{k=1}^{n} P(B \cap A_k)} = \frac{P(B|A_j)P(A_j)}{\sum_{k=1}^{n} P(B|A_k)P(A_k)}$$

Cumulative Distribution Function:

$$F_X(x) = P(s \in S : X(s) \le x) = P(X \le x), -\infty < x < \infty$$

$$P(a < X \le b) = F(b) - F(a)$$

$$P(a \le X \le b) = F(b) - F(a^-)$$

Method	Formula
Sampling with replacement and ordering	n^k
Sampling without replacement and ordering	$P_k^n = \frac{n!}{(n-k)!}$
Sampling without replacement and without ordering	$C_k^n = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$
Sampling with replacement and without ordering	$\binom{n-1+k}{k} = \binom{n-1+k}{n-1}$

Bernoulli Random Variable

$S_X = \{0, 1\}$

$p_0 = q = 1 - p \quad p_1 = p \quad 0 \le p \le 1$

Binomial Random Variable

$S_X = \{0, 1, \ldots, n\}$

$p_k = \binom{n}{k} p^k (1-p)^{n-k} \quad k = 0, 1, \ldots, n$

Geometric Random Variable

First Version: $S_X = \{0, 1, 2, \ldots\}$

$p_k = p(1-p)^k \quad k = 0, 1, \ldots$

Poisson Random Variable

$S_X = \{0, 1, 2, \ldots\}$

$p_k = \frac{\alpha^k}{k!} e^{-\alpha} \quad k = 0, 1, \ldots \text{ and } \alpha > 0$

Uniform Random Variable

$S_X = \{1, 2, \ldots, L\}$

$p_k = \frac{1}{L} \quad k = 1, 2, \ldots, L$

Discrete

Exponential Random Variable

$S_X = [0, \infty)$

$f_X(x) = \lambda e^{-\lambda x} \quad x \ge 0 \text{ and } \lambda > 0$

Uniform Random Variable

$S_X = [a, b]$

$f_X(x) = \frac{1}{b-a} \quad a \le x \le b$

Continuous

Probability Mass Function: $p_X(x) = P(X = x)$

Probability Density Function: $f_X(x) = \frac{dF_X(x)}{dx}$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt \quad P(x < X \le x + \Delta h) \approx f(x)\Delta h$$

$P(a \le x \le b) = \int_a^b f(x)dx$, $F(x) = \int_{-\infty}^{x} f(t)dt$; $\int_{-\infty}^{\infty} f(x)dx = 1$.

Gaussian pdf: $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty$

Gaussian cdf:

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\mu}{\sigma}} e^{-\frac{t^2}{2}} dt = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Function of a Random Variable

Method 1: $F_Y(y) = P(s \in S : Y(s) \le y) = P(s \in S : g(X(s)) \le y)$ and then $f_Y(y) = \frac{dF_Y(y)}{dx}$

Method 2: For all x satisfying $y = g(x)$

$$f_Y(y) = \left.\frac{f_X(x)}{\left|\frac{dy}{dx}\right|}\right|_{\text{All } x_m \text{ satisfying } x = g^{-1}(y)} = \sum_{i=1} \left.\frac{f_X(x)}{\left|\frac{dy}{dx}\right|}\right|_{x = x_i = g^{-1}(y)}$$

1. **(15 points) Easing in to Midterm!**
   (A) (6 points) *My Gift to You*. Circling the correct answer is worth +2 points, circling the incorrect answer is worth -1 point. Not circling either is worth 0 point.
   (a) Suppose $X \sim \mathcal{N}(0,1)$. That is, X is a Gaussian distributed random variable with $\mu = 0$ and $\sigma^2 = 1$. Then, $P(X = 0) = 0.5$.

   TRUE FALSE

   (b) If A and B are mutually exclusive events, $P(A \cup B) = P(A) + P(B) - P(A)P(B)$.

   TRUE FALSE

   (c) When you are sitting on the Santa Monica beach, soaking in the sun on a bright sunny afternoon in winter, and you have no work to do, that is the best time to solve the Probability problems.

   TRUE FALSE (No Wrong Answer)

   (B) (2 points) Suppose $P(A) = \frac{1}{3}$, $P(A \cup B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{5}$. Find $P(B)$.
   Solution:
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$P(B) = P(A \cup B) + P(A \cap B) - P(A) = \frac{1}{2} + \frac{1}{5} - \frac{1}{3} = \frac{11}{30}$$

   (C) (7 points) It is known that for two events, A and B, $P(A) = \frac{4}{5}$ and $P(B|A) = \frac{1}{2}$, and that A and B are independent. For each of the following statements, say *true*, *false* or *cannot tell* and justify your answer.
   (a) (2 points) A and B are mutually exclusive.
   (b) (2 points) A and $A \cap B$ are independent.
   (c) (2 points) $P(B) = P(A|B)$.
   (d) (1 point) $P(B) \leq P(A)$.
   Solution:
   (a) A and B are mutually exclusive.
   False: $P(A \cap B) = P(A)P(B)$. But because A and B are independent $P(B) = P(B|A) = \frac{1}{2}$. So $P(A \cap B) = \frac{2}{5} \neq 0$.
   (b) A and $A \cap B$ are independent.
   False: $P\big(A \cap (A \cap B)\big) = P(A \cap B) = \frac{2}{5} \neq P(A)P(A \cap B) = \frac{8}{25}$.
   (c) $P(B) = P(A|B)$
   False: $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{4}{5}$, while $P(B) = \frac{1}{2}$.
   (d) $P(B) \leq P(A)$
   True: $P(B) = \frac{1}{2}$ and $P(A) = \frac{4}{5}$.

2. **(9 points) Let's do some counting and apply Bayes**
   (A) (4 points) 52-card deck consists of 4 suits (clubs, diamonds, hearts, spades), and each suit has 13 cards (2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A). Suppose you are dealt a poker hand consisting of 5 cards. Compute the probability of getting flush. Flush hand is 5 cards of the same suit. Leave your answer in the form of $\binom{n}{k}$.

Solution:

$$P(flush) = \frac{4\binom{13}{5}}{\binom{52}{5}} = 0.002$$

(B) (5 points) *Binary Non-Symmetric Channel*. Consider a `binary non-symmetric channel' with input of '0' and '1' each with probability of 0.5. Suppose the channel transition probabilities are: $P(0 \text{ received}|0 \text{ sent}) = 0.95$; $P(1 \text{ received}|0 \text{ sent}) = 0.05$; $P(1 \text{ received}|1 \text{ sent}) = 0.90$; $P(0 \text{ received}|1 \text{ sent}) = 0.10$. Suppose we received a '1', what is the probability that a '1' was sent?

Solution:

Applying Bayes Rule and Theorem of Total Probability,

$$P(1 \text{ sent}|1 \text{ received}) = \frac{P(1 \text{ received}, 1 \text{ sent})}{P(1 \text{ received})} = \frac{P(1 \text{ received}|1 \text{ sent})P(1 \text{ sent})}{P(1 \text{ received})}$$
$$= \frac{0.90 \times 0.5}{P(1 \text{ received}|1 \text{ sent})P(1 \text{ sent}) + P(1 \text{ received}|0 \text{ sent})P(0 \text{ sent})}$$
$$= \frac{0.9 \times 0.5}{0.9 \times 0.5 + 0.05 \times 0.5} = \frac{0.45}{0.475} = 0.9474$$

3. (11 points) Binomial and Uniform Random Variables

(A) (6 points) A wheel used in gambling can stop in 18 equally likely positions numbered from 1 to 18. A person places bets on all the positions divisible by 3.

(a) (4 points) What is the probability that the person will win exactly three times in 8 attempts? Leave your answer in the form of $\binom{n}{k}$.

(b) (2 points) Can we approximate using Poisson random variable? Why or why not?

Solution:

(a) Since there are 6 possible number from 1 to 18 divisible by 3, and each one is equally likely, then the probability of winning at each attempt is $p = \frac{6}{18} = \frac{1}{3}$. The probability of losing at each attempt is $q = \frac{2}{3}$. Since this is a Bernoulli trial of length $n = 8$; then $p(X = 3) = C_3^8 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^5$

(b) No because $n = 8, p = \frac{1}{3}$, and $np = \frac{8}{3}$ does not satisfy $n \gg np \gg p$.

(B) (4+1 points) Let X be uniformly distributed on $(0, 1)$. Find cdf and pdf of $Y = -\frac{1}{a}\ln(1 - X)$, where a > 0. What is the name of this random variable ?

Solution:

$$P(Y \le y) = P\left(-\frac{1}{a}\ln(1 - X) \le y\right) = P(\ln(1 - X) \ge -ay) = P(1 - X \ge e^{-ay})$$
$$= P(X \le 1 - e^{-ay}) = F_X(1 - e^{-ay}) = 1 - e^{-ay} \quad for\ y > 0$$

Therefore,

$$F_Y(y) = 1 - e^{-ay} \quad for\ y > 0$$

$$f_Y(y) = \begin{cases} 0, & otherwise \\ ae^{-ay}, & if\ y > 0 \end{cases}$$

Exponential random variable.

4. (15 points) Fun with Rayleigh and Gaussian Distributions

(A) (3.5+1.5 points) The Rayleigh random variable has cdf

$$F_R(r) = \begin{cases} 0, & r < 0 \\ 1 - e^{-\frac{r^2}{2\sigma^2}}, & r \geq 0 \end{cases}$$

Find $P(\sigma < R \leq 2\sigma)$. Find pdf of this random variable.

Solution:

$$P(\sigma < R \leq 2\sigma) = F_R(2\sigma) - F_R(\sigma) = \left(1 - e^{-\frac{4\sigma^2}{2\sigma^2}}\right) - \left(1 - e^{-\frac{\sigma^2}{2\sigma^2}}\right) = e^{-\frac{1}{2}} - e^{-2}$$

$$f_R(r) = \begin{cases} 0, & r < 0 \\ \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}, & r \geq 0 \end{cases}$$

(B) (10 points) Suppose the length of a rod is modelled by a Gaussian random variable of $\mu = 1$ and $\sigma^2 = 0.25$. But this model clearly cannot be completely correct since a Gaussian random variable can take on negative values, which is not possible for the length of the rod. Thus, define a new random variable X to be the random variable whose pdf is that of the original Gaussian random variable defined only for positive-values and normalized for unit area. (Note: In this model, we let the length be arbitrarily long, but with very low probability.) Denote this pdf by $f_X(x), x \geq 0$; $f_X(x) = 0, x < 0$. (Note, for this new pdf $\int_0^\infty f_X(x)dx = 1$.) In addition, you are given a very small table of the Gaussian cdf $\Phi(x)$. Note: $\Phi(x) = 1 - Q(x)$.

$\Phi(-3) = 0.00135$; $\Phi(-2) = 0.0228$; $\Phi(-1) = 0.159$; $\Phi(-0.3) = 0.382$; $\Phi(-0.2) = 0.421$; $\Phi(-0.1) = 0.460$.

(a) (6.5 points) Write down explicitly the pdf $f_X(x)$ for $x \geq 0$
[*Hint:* Let Y be the original Gaussian random variable with μ and σ^2 as given. To get the new random variable X from Y, it is already given that if follows distribution of Y but only from $(0, \infty)$. So the challenge here is to find a new pdf $\int_0^\infty f_X(x)dx = 1$. To ensure this, you have to find the area under the pdf of Y on the right side of the origin and use that to normalize to get unit area under $f_X(x)$. That is, if area under right side of origin of $f_Y(y)$, $\int_0^\infty f_Y(y)dy = a$, then $f_X(x) = \frac{1}{a}\int_0^\infty f_Y(y)dy$. Also, note that $\int_0^\infty f_Y(y)dy = 1 - \int_{-\infty}^0 f_Y(y)dy$.]

(b) (3.5 points) Find the probability the rod modelled by this X is between 0.9 and 1.1 explicitly.

Solution:

(a) Consider a Gaussian pdf of $f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}, -\infty < y < \infty$, where $\mu = 1$ and $\sigma^2 = 0.25$. Our new random variable follows distribution of y but for only positive values. So, in order to satisfy $\int_0^\infty f_X(x)dx = 1$, we need to normalize it for the area lying under the curve $\int_0^\infty f_Y(y)dy$. So let's calculate the area under the curve $\int_0^\infty f_Y(y)dy$. Area on the left side of the origin is $\int_{-\infty}^0 f_Y(y)dy = \Phi\left(\frac{0-1}{\sqrt{0.25}}\right) = \Phi(-2) = 0.0228$. In other words, the area to the right of the origin of $f(y)$ contains only the probability of 0.9772. Thus,

$$f_X(x) = \frac{1}{0.9772}[(2\pi\sigma^2)^{-0.5} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, 0 \leq x < \infty$$

(b) $P(0.9 \leq X \leq 1.1) = \int_{0.9}^{1.1} f_X(x)dx = \frac{1}{0.9772}\left[\Phi\left(\frac{1.1-1}{\sqrt{0.25}}\right) - \Phi\left(\frac{0.9-1}{\sqrt{0.25}}\right)\right] = \frac{1}{0.9772}[\Phi(0.2) - \Phi(-0.2)] = \frac{1}{0.9772}[1 - \Phi(-0.2) - \Phi(-0.2)] = \frac{0.158}{0.9772} = 0.1617.$