EE 131A
Probability
Instructor: Lara Dolecek

Winter 2020 Midterm B
Monday, February 3, 2020

Maximum score is 100 points. You have 110 minutes to complete the exam. Please show your work. Good luck!

Your Na [redacted] right:

Problem	Score	Possible
1	10	10
2	15	15
3	15	15
4	15	15
5	13	15
6	15	15
7	15	15
Total	98	100

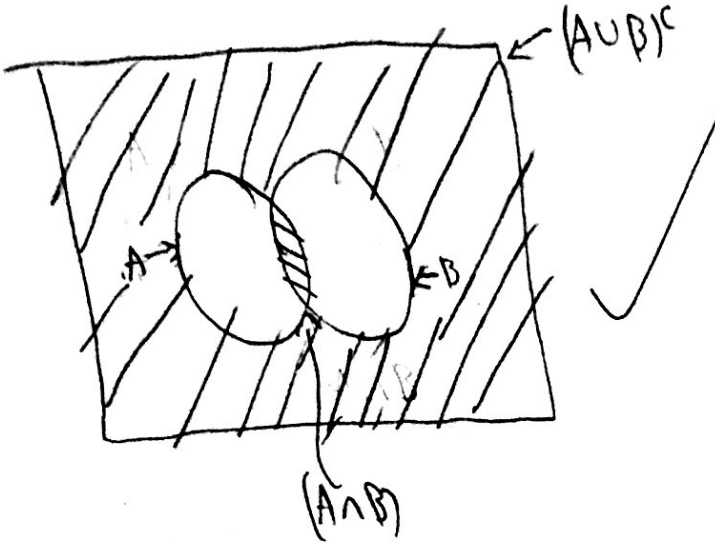
10

1. (5+5 pts)

(a) Draw a Venn Diagram for the events A and B. Shade in the area corresponding to the events $(A \cap B)$ and $(A \cup B)^c$. Clearly indicate which area belongs to which event.

(b) Assume that we throw a single six-sided die. Let A be the event that the result is even and let B be the event that the result is less than or equal to 3. What is $P(A \cap B)$ and $P(A \cup B)$?

a)



b) Sample space for A: $\{2, 4, 6\}$

Sample space for B: $\{1, 2, 3\}$

Sample space for $A \cap B$ (i.e. even AND ≤ 3): $\{2\}$

Sample space for $A \cup B$: $\{1, 2, 3, 4, 6\}$

$$\boxed{\begin{array}{l} P(A \cap B) = \frac{1}{6} \\ P(A \cup B) = \frac{5}{6} \end{array}}$$

15

2. (3+5+7 pts) Assume there are 5 jars numbered 1 to 5. The ith jar contains i black balls, $6-i$ red balls, and 7 green balls. A jar is selected uniformly at random and a ball is selected from that jar. Let the events B, R, and G represent the events that a black, red, or green ball is chosen, respectively. Let A_k represent the event that the kth jar is chosen.

(a) What is $P(B|A_k)$? Write the answer in terms of k.

(b) What is $P(G)$, $P(B)$, and $P(R)$?

(c) Given that the ball selected was black, what is the probability that the ball came from the kth jar, i.e. $P(A_k|B)$? Write the answer in terms of k.

a) # of balls in a jar: $i+6-i+7 = 13$ balls

k black balls in jar A_k

$$\boxed{P(B|A_k) = \frac{k}{13}}$$ ✓

b) $P(G) = \sum_1^5 P(G|A_k)\cdot P(A_k)$

$= \frac{1}{5}\left(\frac{7}{13}+\frac{7}{13}+\frac{7}{13}+\frac{7}{13}+\frac{7}{13}\right)$

$= \frac{1}{5}\left(\frac{7}{13}\right)\cdot 5$

$= \frac{7}{13}$

$P(B) = \sum_1^5 P(B|A_k)\cdot P(A_k)$

$= \frac{1}{5}\left(\frac{1}{13}+\frac{2}{13}+\frac{3}{13}+\frac{4}{13}+\frac{5}{13}\right)$

$= \frac{1}{5}\left(\frac{15}{13}\right) = \frac{3}{13}$

b) (continued)

$P(R) = \sum_1^5 P(R|A_k)\cdot P(A_k)$

$= \frac{1}{5}\left(\frac{5}{13}+\frac{4}{13}+\frac{3}{13}+\frac{2}{13}+\frac{1}{13}\right)$

$= \frac{1}{5}\left(\frac{15}{13}\right) = \frac{3}{13}$

$$\boxed{\begin{aligned} P(G) &= \tfrac{7}{13} \\ P(B) &= \tfrac{3}{13} \\ P(R) &= \tfrac{3}{13} \end{aligned}}$$ ✓ ✓ ✓

c) Bayes' Rule:

$P(A_k|B) = \frac{P(B|A_k)\cdot P(A_k)}{P(B)} = \frac{\left(\frac{k}{13}\right)\cdot\left(\frac{1}{5}\right)}{\frac{3}{13}}$

$= \frac{\left(\frac{k}{5}\right)}{3} = \frac{k}{15}$

$$\boxed{P(A_k|B) = \frac{k}{15}}$$ ✓

15

3. (15 pts) We select two distinct numbers (a, b) in the range 1 to 99 (inclusive). How many ways can we pick a and b such that their sum is even and a is a multiple of 9?

11 ways to select a

6 odd a's

For odd a, we want an odd b for a+b to be even:

50 odd b's

Subtract 1 (because a and b are distinct):

49 odd b's

$$\begin{array}{r} \overset{5}{49} \\ \times 6 \\ \hline 294 \end{array}$$

5 even a's

For even a, we want an even b for a+b to be even:

49 even b's

Subtract 1 (because a and b are distinct):

48 even b's

$$\begin{array}{r} \overset{4}{48} \\ \times 5 \\ \hline 240 \end{array}$$

$$\begin{array}{r} \overset{1}{294} \\ +240 \\ \hline 534 \end{array}$$

$\boxed{534 \text{ ways}}$ ✓

Note: I'm assuming that "two distinct numbers (a,b)" means that $a \neq b$.

15

4. (15 pts) True or False.

Circling the correct answer is worth +3 points, circling the incorrect answer is worth −1 points. Not circling either is worth 0 points.

(a) A discrete random variable has jump discontinuities in its cumulative distribution function.

TRUE FALSE

(b) $P(A|B) = P(B|A)$ holds if and only if the events A and B are mutually exclusive.

TRUE FALSE

should be independant

(c) For a random variable X, $\text{VAR}(aX) = a\text{VAR}(X)$ for all real values of a.

TRUE FALSE

(d) For random variables X and Y, if $\text{E}[XY] = \text{E}[X]\text{E}[Y]$, then X and Y are uncorrelated.

TRUE FALSE

definition of uncorrelated

(e) If events X and Y are mutually exclusive, then they are also independent.

TRUE FALSE

13

5. (~~15 pts~~) Let A and B be two events. Given that $P(B) > 0$, prove

$$P(A \cup B^c | B) = P(A \cap B | B).$$

You may use any result taught in lecture or homework.

$$P(A \cup B^c | B) = \frac{P((A \cup B^c) \cap B)}{P(B)}$$

$(A \cup B^c) \cap B = (A \cap B) \cup (B^c \cap B)$ (distributivity)

$= A \cap B$ $\quad (B^c \cap B = \emptyset$ and $(A \cap B) \cup \emptyset = A \cap B)$

$$P(A \cup B^c | B) = \frac{P(A \cap B)}{P(B)}$$

Missing step (-2)

$$P(A \cap B | B) = \frac{P(A \cap B)}{P(B)} = P(A \cup B^c | B)$$

Thus $P(A \cup B^c | B) = P(A \cap B | B)$

15

6. (3+7+5 pts) Suppose that the continuous random variable X has pdf

$$f(x) = \begin{cases} cx^2 & |x| \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find c such that the pdf is valid.

(b) Find E[X] and VAR(X).

(c) Find $P(X \ge \frac{1}{2})$.

a) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int_{-1}^{1} cx^2\,dx = 1$

$c\left(\frac{x^3}{3}\right)\Big|_{-1}^{1} = 1$

$c\left(\frac{1}{3} + \frac{1}{3}\right) = 1$

$c\left(\frac{2}{3}\right) = 1$

$\boxed{c = \frac{3}{2}}$

b) $E[x] = \int_{-\infty}^{\infty} t\,f(t)\,dt$

$E[x] = \int_{-1}^{1} t\left(\frac{3}{2}\right)t^2\,dt$

$= \frac{3}{2}\int_{-1}^{1} t^3\,dt$

$= \frac{3}{2}\left(\frac{t^4}{4}\right)\Big|_{-1}^{1}$

$= \frac{3}{2}\left(\frac{1}{4} - \frac{1}{4}\right)$

$= 0$

$\boxed{E[x] = 0}$

b) (continued)

$VAR(X) = E[x^2] - m_x^2$

$m_x = E[x] = 0$

$E[x^2] = \int_{-\infty}^{\infty} t^2 f(t)\,dt$

$= \int_{-1}^{1} t^2\left(\frac{3}{2}\right)t^2\,dt$

$= \frac{3}{2}\int_{-1}^{1} t^4\,dt$

$= \frac{3}{2}\left(\frac{t^5}{5}\right)\Big|_{-1}^{1}$

$= \frac{3}{2}\left(\frac{1}{5} + \frac{1}{5}\right)$

$= \frac{3}{2}\left(\frac{2}{5}\right) = \frac{3}{5}$

$VAR(X) = \frac{3}{5} - 0^2$

$\boxed{VAR(X) = \frac{3}{5}}$

15

7. (7+8 pts) Let $X_1, X_2, \ldots, X_n$ be independent Bernoulli random variables with parameters p_1 for even i and p_2 for odd i where $p_2 \neq p_1$ and n is a positive even number. Let Y be the sum of all the X_i's.

(a) Is Y a Binomial Random Variable? Justify your answer.

(b) Compute E[Y].

~~a) No, Y is not a Binomial Variable.~~
~~Binomial Random Variable,~~

a) No, Y is not a Binomial Random Variable. Part of the definition of a Binomial Random Variable is that each of the independent Bernoulli random variables has the same probability of success; however, that is not the case here, because X_i has success probability p_1 for even i and p_2 for odd i, i.e. the independent Bernoulli random variables do not all have the same probability of success.

7

b) $E[Y] = E\left[\sum_{k=1}^{n} X_k\right] = \sum_{k=1}^{n} E[X_k] = \sum_{m=1}^{\frac{n}{2}} E[X_{2m-1}] + \sum_{m=1}^{\frac{n}{2}} E[X_{2m}]$ (separate into sums of odd subscript RVs and even subscript RVs)

4

$= \frac{n}{2}(p_2) + \frac{n}{2}(p_1) = \boxed{\frac{n(p_1+p_2)}{2}}$

(Note: for a Bernoulli random variable X with success probability p, $E[X] = p$)

8