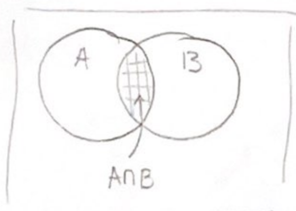
10

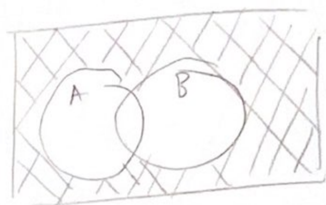
1. (5+5 pts)

   (a) Draw a Venn Diagram for the events A and B. Shade in the area corresponding to the events $(A \cap B)$ and $(A \cup B)^c$. Clearly indicate which area belongs to which event.

   (b) Assume that we throw a single six-sided die. Let A be the event that the result is even and let B be the event that the result is less than or equal to 3. What is $P(A \cap B)$ and $P(A \cup B)$?

a)





$(A \cup B)^c$

$A^c \cap B^c$

✓

b) A → even (2, 4, 6)

B → less than or equal to 3 → (1, 2, 3)

$P(A \cap B) = \frac{1}{6}$

✓

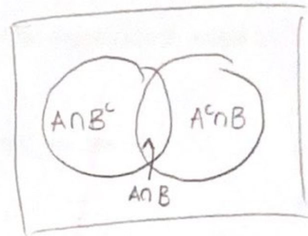
$P(A \cup B) = \frac{5}{6}$

15

2. (15 pts) Let A and B be two events. Given that $P(B) > 0$, prove

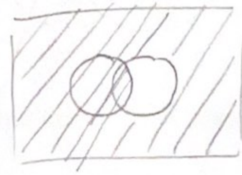
$$P(A \cup B^c | B) = P(A \cap B | B).$$

You may use any result taught in lecture or homework.



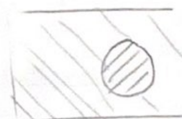
$$P(A \cup B^c | B)$$

$$= \frac{P[(A\cup B^c)\cap B)}{P(B)}$$



$A \cup B^c$

$$= \frac{P[(A\cap B)\cup \overset{\phi}{(B^c\cap B)}]}{P(B)}$$



$B^c \cap B = \phi$

$$= \frac{P(A\cap B) + \overset{0}{P(B^c\cap B)}}{P(B)} = \frac{P(A\cap B)}{P(B)} = P(A\cap B|B)$$

8

3. (15 pts) True or False.

Circling the correct answer is worth +3 points, circling the incorrect answer is worth −1 points. Not circling either is worth 0 points.

(a) For a random variable X, $\mathrm{VAR}(aX) = a\mathrm{VAR}(X)$ for all real values of a.

+3 TRUE FALSE

(b) $P(A|B) = P(B|A)$ holds if and only if the events A and B are mutually exclusive.

TRUE FALSE

(c) A discrete random variable has jump discontinuities in its cumulative distribution function.

+3 TRUE FALSE

(d) If events X and Y are mutually exclusive, then they are also independent.

−1 TRUE FALSE if independent not ness mutually excl.

(e) For random variables X and Y, if $\mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y]$, then X and Y are uncorrelated.

+3 TRUE FALSE

55 (11
5

15

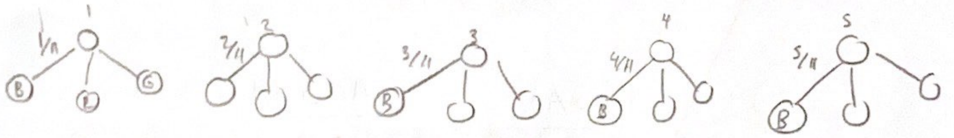
4. (3+5+7 pts) Assume there are 5 jars numbered 1 to 5. The ith jar contains i black balls, $6 - i$ red balls, and 5 green balls. A jar is selected uniformly at random and a ball is selected from that jar. Let the events B, R, and G represent the events that a black, red, or green ball is chosen, respectively. Let A_k represent the event that the kth jar is chosen.

   (a) What is $P(B|A_k)$? Write the answer in terms of k. $\quad P(A_k) = 1/5$

   (b) What is $P(G)$, $P(B)$, and $P(R)$?

   (c) Given that the ball selected was black, what is the probability that the ball came from the kth jar, i.e. $P(A_k|B)$? Write the answer in terms of k.

a) $P(B|A_k) \rightarrow$ prob. of getting a black ball given that you chose the kth jar



$$\boxed{P(B|A_k) = \frac{k}{11}}$$

b) $P(G) = P(G|A_1)\cdot P(A_1) + P(G|A_2)P(A_2) + \cdots + P(G|A_5)\cdot P(A_5)$

$$P(G) = \frac{1}{5}\left[\frac{5}{11} + \frac{5}{11} + \frac{5}{11} + \frac{5}{11} + \frac{5}{11}\right] = \frac{1}{5}\left[\frac{25}{11}\right] = 5/11$$

$P(B) = P(B|A_1)\cdot P(A_1) + P(B|A_2)\cdot P(A_2) + P(B|A_3)\cdot P(A_3) + P(B|A_4)\cdot P(A_4) + P(B|A_5)\cdot P(A_5)$

$$\frac{k}{(6-k)+k+5} = \frac{k}{11}$$

$$P(B) = \frac{1}{5}\left[\frac{1}{11} + \frac{2}{11} + \frac{3}{11} + \frac{4}{11} + \frac{5}{11}\right] = \frac{1}{5}\left[\frac{15}{11}\right] = \frac{3}{11}$$

$P(R) = P(R|A_1)\cdot P(A_1) + P(R|A_2)P(A_2) + \cdots + P(R|A_5)P(A_5)$

$P(G)+P(B)+P(R) = 1$

$$= \frac{1}{5}\left[\frac{5}{11} + \frac{4}{11} + \frac{3}{11} + \frac{2}{11} + \frac{1}{11}\right] = \frac{1}{5}\left[\frac{15}{11}\right] = \frac{3}{11}$$

$P(A_k|B)$ = given that the ball is black, probability of coming from A_kth jar

$$= \frac{P(B|A_k)P(A_k)}{P(B)} = \frac{(k/11)(1/5)}{3/11} = \frac{\frac{k}{55}}{\frac{3}{11}} = \frac{11k}{55\cdot 3} = \frac{k}{15}$$

15

0 - 99 = 100
1 - 99 = 99

5. (~~15~~ pts) We select two distinct numbers (a, b) in the range 1 to 99 (inclusive). How many ways can we pick a and b such that their sum is even and a is a multiple of 9?

w/o rep w/o order

a is a multiple of 9 → { 9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99

o e o e o e o e o e o

a + b must be even

odd + odd = even
even + even = even
even + odd = odd
odd + even = odd

I know this, but I also don't know

5 → even mult of 9
6 → odd mult of 9

o e o e ... e o
1 2 3 4 ... 98 99

same even
same odd

odds = even + 1

99 total num

99 = e + e + 1

98 = 2e

$e = \frac{98}{2} = 49$ even

50 odds

a → ways to pick an odd mult of 9 $\binom{6}{1}$ · ways to pick an even mult of 9 $\binom{5}{1}$

b → ways to pick an odd from the whole pool of num $\binom{50-1}{1}$ · ways to pick an even from pool of num left $\binom{49-1}{1}$

total num of options = (6)(49) + (5)(48) = 294 + 240 = 534 ✓

options of pick a

49 × 6 = 294

48 × 5 = 240

15

6. (7+8 pts) Let $X_1, X_2, \ldots, X_n$ be independent Bernoulli random variables with parameters p_1 for even i and p_2 for odd i where $p_2 \neq p_1$ and n is a positive even number. Let Y be the sum of all the X_i's.

   (a) Is Y a Binomial Random Variable? Justify your answer.

   (b) Compute E[Y].

a) A binomial RV takes on parameters n and p n is the number of independent Bernoulli RV we have where they all have probability p

Since p varies on RV depending on even or odd we cannot say that the sum of X is a binomial RV

b) $E[Y] \Rightarrow E\left[\sum_i X_e + \sum_i X_o\right] = E[X_e] + E[X_o]$

now these are binomials of its own

n even → same num of odds as even

$E[Y] = \frac{n}{2} p_1 + \frac{n}{2} p_2$

$E[Y] = \frac{n}{2}(p_1 + p_2)$

8

15

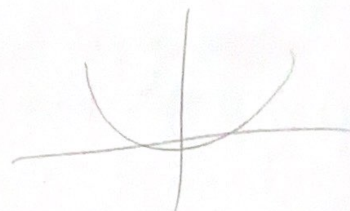
7. (3+7+5 pts) Suppose that the continuous random variable X has pdf

$$f(x) = \begin{cases} cx^2 & |x| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find c such that the pdf is valid.

(b) Find E[X] and VAR(X).

(c) Find $P(X \geq \frac{1}{2})$.



a) $\int_{-\infty}^{\infty} cx^2\,dx = 1 \rightarrow \int_{-1}^{1} cx^2\,dx = 1$

$= c\left[\frac{1}{3}x^3 \Big|_{-1}^{1}\right] = 1 \rightarrow c\left[\frac{1}{3}(1)^3 - \frac{1}{3}(-1)^3\right] = 1$

$c\left[\frac{1}{3} + \frac{1}{3}\right] = 1 \rightarrow \frac{c \cdot 2}{3} = 1$ $\quad c = \frac{3}{2}$

b) $\mathbb{E}[X] = \int_{-\infty}^{\infty} x\, cx^2\,dx = \int_{-1}^{1} cx^3\,dx \rightarrow \frac{3}{2}\left[\frac{1}{4}x^4 \Big|_{-1}^{1}\right] = \frac{3}{2}\left[\frac{1}{4} - \frac{1}{4}\right] = 0$

$VAR[X] \Rightarrow \mathbb{E}[x^2] = \int_{-1}^{1} cx^4\,dx \rightarrow \frac{3}{2}\left[\frac{1}{5}x^5 \Big|_{-1}^{1}\right] = \frac{3}{2}\left[\frac{1}{5} + \frac{1}{5}\right] = \frac{3}{\cancel{2}} \cdot \frac{\cancel{2}}{5}$

$\mathbb{E}[X^2] = 3/5$

$VAR[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 3/5 - 0 = 3/5$

c) $P(X \geq 1/2) = 1 - F_X(1/2) + \cancel{F_X(1/2)}^{\to 0}$

$= 1 - F_X(1/2)$

$$F_x = \int_{-1}^{a} \frac{3}{2} x^2 dx \Rightarrow \frac{3}{2}\left[\frac{1}{3}x^3 \Big|_{-1}^{a}\right]$$

$$= \frac{3}{2}\left[\frac{1}{3}a^3 - \frac{(-1)^3}{3}\right] = \frac{\cancel{3}}{2}\left[\frac{a^3+1}{\cancel{3}}\right] = \frac{a^3+1}{2}$$

$$F_x(1/2) = \frac{(1/2)^3+1}{2} = \frac{\frac{1}{8}+\frac{8}{8}}{\frac{2}{1}} = \frac{\frac{9}{8}}{\frac{2}{1}} = \frac{9}{16}$$

$$P(X \geq 1/2) = 1 - 9/16$$

$$= \frac{16}{16} - \frac{9}{16} = \frac{7}{16}$$