EE 131A
Probability
Instructor: Lara Dolecek

Winter 2018 Midterm
Tuesday, February 12, 2018

Maximum score is 100 points. You have 110 minutes to complete the exam. Please show your work. Good luck!

Your Name:

[redacted]

Your ID Number:

[redacted]

Name of person on your left:

[redacted]

Name of person on your right:

(Aisle Seat)

Problem	Score	Possible
1	10	10
2	10	10
3	13	15
4	15	15
5	8	10
6	10	10
7	15	15
8	8	15
Total	89	100

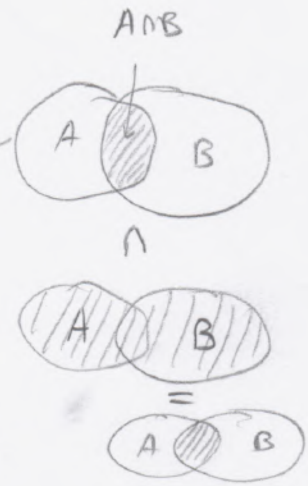
10

1. (10 pts Show that if $P(A) > 0$, then

$$P(A \cap B|A) \geq P(A \cap B|A \cup B)$$

$$\Rightarrow \quad P(A\cap B|A) = \frac{P((A\cap B)\cap A)}{P(A)} = \frac{P(A\cap B)}{P(A)}$$

$$\Rightarrow \quad P(A\cap B|A\cup B) = \frac{P((A\cap B)\cap(A\cup B))}{P(A\cup B)} = \frac{P(A\cap B)}{P(A\cup B)}$$



We know that $P(A\cup B) > P(A)$, where equality holds if $B = A$ or $B = \phi$ Thus, we know that

$$\frac{P(A\cap B)}{P(A)} \geq \frac{P(A\cap B)}{P(A\cup B)}, \quad \text{meaning}$$

$$\boxed{P(A\cap B|A) \geq P(A\cap B|A\cup B)}$$

+10

10

2. 4+6 pts) Let X be a random variable that takes integer values from 0 to 9 with equal probability $\frac{1}{10}$

a) Find the PMF of the random variable $Y = X \bmod 3$.

(b Find the PMF of the random variable $Y = 5 \bmod (X + 1$

a) $S_Y = \{0, 1, 2\}$

$$P(Y=0) = P(X=0) + P(X=3) + P(X=6) + P(X=9)$$
$$= 4\left(\tfrac{1}{10}\right) = \boxed{\tfrac{2}{5}}$$

$$P(Y=1) = P(X=1) + P(X=4) + P(X=7)$$
$$= 3 \cdot \left(\tfrac{1}{10}\right) = \boxed{\tfrac{3}{10}}$$

$$P(Y=2) = P(X=2) + P(X=5) + P(X=8)$$
$$= 3 \cdot \left(\tfrac{1}{10}\right) = \boxed{\tfrac{3}{10}}$$

$R = 0, 1, 2, 1, 0, 5, 5, 5, 5, 5$

b) $S_Y = \{0, 1, 2, 5\}$, $\quad X+1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$$P(Y=0) = P(X+1=5) + P(X+1=1) = 2\,\tfrac{1}{10} = \boxed{\tfrac{1}{5}}$$

$$P(Y=1) = P(X+1=2) + P(X+1=4) = 2\,\tfrac{1}{10} = \boxed{\tfrac{1}{5}}$$

$$P(Y=2) = P(X+1=3) = \boxed{\tfrac{1}{10}}$$

$$P(Y=5) = P(X+1=6) + P(X+1=7) + P(X+1=8) + P(X+1=9) + P(X+1=10)$$
$$= 5 \cdot \tfrac{1}{10} = \boxed{\tfrac{1}{2}}$$

+10

13

3. (7+8 pts

a) A police department in a small city consists of 10 officers. If the department policy is to have 5 of the officers patrolling the streets, 2 of the officers working full time at the station, and 3 of the officers on reserve at the station, how many different divisions of the 10 officers into the 3 groups are possible?

(b A 5-card hand is dealt from a well-shuffled deck of 52 playing cards. What is the probability that the hand contains at least one card from each of the four suits?

a) _ _ _ _ _ | _ _ _ | _ _

$$\boxed{\frac{10!}{5!\,3!\,2!}}$$

+7

b) # of combinations w/ one of each suit

$$\binom{13}{1}\binom{13}{1}\binom{13}{1}\binom{13}{1}\binom{52-4}{1} = 13^4\,48$$

Choose 1 Card from Suit 1

Choose 1 from suit 2

Choose 1 from suit 3

Choose 1 from suit 4

Choose any leftover card

Total # of Card combinations:

$$\binom{52}{5}$$

Probability

$$\boxed{\frac{13^4\,48}{\binom{52}{5}}}$$

+6

15

4. (15 pts True or False.

Circling the correct answer is worth +3 points, circling the incorrect answer is worth −1 points. Not circling either is worth 0 points.

a) The expected value of a sum of random variables is equal to the sum of the expected values of each random variable.

TRUE FALSE

(b Discrete variables have means that are always integer values.

TRUE FALSE

c The probability of the success of a trial or observation for a binomial probability distribution depends on the trial or observation that came before it.

TRUE FALSE

d) If events X and Y are independent, then they are also mutually exclusive.

TRUE FALSE

e If events X and Y are independent, $Var[X] = a$, and $Var[Y] = b$, then $Var[a + b] = Var[X] + Var[Y]$ is always true.

TRUE FALSE

8 ☆

5. (5+5 pts

a) Prove the memoryless property of geometric random variables.

(b The number of years a radio functions is exponentially distributed with parameter $\lambda = \frac{1}{8}$. If David bought a functional radio which has been used for 8 years, what is the probability that it will be working after an additional 8 years?

a) Let X be a geometric random variable

Let p be the probability of a specific event A

Let $P(X=k)$ be probability of achieving event A on k^{th} trial

$$\Rightarrow P(X=k) = p(1-p)^{k-1}$$

$$\rightarrow P[X \geq k+j \mid X>j] = \frac{P[\{X \geq k+j\} \cap \{X>j\}]}{P[X>j]} = \frac{\quad}{1 - \sum_{k=1}^{j} p(1-p)^{j-1}}$$

+3

$$= \boxed{P(X \geq k)}$$

b) $f_X(x) = \begin{cases} \frac{1}{8} e^{-\frac{1}{8}x} & , x \geq 0 \\ 0 & x < 0 \end{cases}$ X = # of years radio functions

Exponential RV is memoryless

$$P(X \geq 8+8 \mid X>8) = P(X \geq 8)$$

$$= 1 - P(X \leq 8) + P(X=8)$$

$$= 1 - F_X(8) + 0$$

$$= 1 - (1 - e^{-1/8(8)}) + 0$$

$$= \boxed{e^{-1}}$$

+5

$$F_X = \begin{cases} 1 - e^{-1/8 x} & , x \geq 0 \\ 0 & , \text{else} \end{cases}$$

10

6. (3+3+4 pts) Suppose that the continuous random variable X has pdf

$$f(x = \begin{cases} c(1-x^2) & -1 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find c such that the pdf is valid.

(b Find the expected value of X

c Find the variance of X

a) $\int_{-\infty}^{\infty} f_x(x)\,dx = \int_{-1}^{1} c(1-x^2)\,dx = c\Big|_{-1}^{1} x - \frac{x^3}{3}$

$= c\left[\left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right)\right]$

$= c\left[\frac{2}{3} - \left(\frac{-2}{3}\right)\right] = c\left(\frac{4}{3}\right) = 1 \Rightarrow \boxed{c = \frac{3}{4}}$

b) $E[X] = \int_{-\infty}^{\infty} x f_x(x)\,dx = \int_{-1}^{1} \frac{3}{4} x(1-x^2)\,dx$

$= \frac{3}{4}\int_{-1}^{1} x - x^3\,dx = \frac{3}{4}\Big|_{-1}^{1} \frac{x^2}{2} - \frac{x^4}{4}$

$= \frac{3}{4}\left[\left(\frac{1}{2}-\frac{1}{4}\right) - \left(\frac{1}{2}-\frac{1}{4}\right)\right] = \boxed{0}$

c) $E[X^2] = \int_{-\infty}^{\infty} x^2 f_x(x)\,dx = \int_{-1}^{1} \frac{3}{4} x^2(1-x^2)\,dx$

$= \frac{3}{4}\int_{-1}^{1} x^2 - x^4\,dx$

$= \frac{3}{4}\Big|_{-1}^{1} \frac{x^3}{3} - \frac{x^5}{5}$

$= \frac{3}{4}\left[\left(\frac{1}{3}-\frac{1}{5}\right) - \left(\frac{-1}{3}+\frac{1}{5}\right)\right]$

15

7 (7+8 pts

a) Find the characteristic function of the uniform continuous random variable, distributed uniformly on the interval $-b, b]$.

(b Find the mean and variance of X by applying the moment theorem.

$$f_X(x) = \begin{cases} \frac{1}{2b}, & -b \le x \le b \\ 0, & \text{else} \end{cases}$$

a) $$\Phi_X(w) = \int_{-\infty}^{\infty} f_X(x) e^{jwx}\, dx$$

$$= \int_{-b}^{b} \frac{1}{2b} e^{jwx}\, dx$$

At $w=0$

$$\Rightarrow \Phi_X(w) = \int_{-b}^{b} \frac{1}{2b} e^{j(0)(x)}\, dx = \Big|_{-b}^{b} \frac{x}{2b} = \frac{b-(-b)}{2b} = 1$$

At $w \neq 0$, (Euler's identity)

$$\Rightarrow \Phi_X(w) = \frac{1}{2b} \Big|_{-b}^{b} \frac{1}{jw} e^{jwx} = \frac{1}{2bjw}\left(e^{jbw} - e^{-jbw}\right) = \frac{1}{bw} \sin(bw)$$

$$\boxed{\Phi_X(w) = \begin{cases} 1, & w=0 \\ \frac{\sin(bw)}{bw}, & w \neq 0 \end{cases}}$$

← Note, This is continuous because $\lim_{w\to 0} \frac{\sin(bw)}{bw} = \lim_{w \to 0} \frac{b\cos(bw)}{b} = \frac{b}{b} = 1$

b) $$\frac{d\Phi_X(w)}{dw} = \frac{1}{b}\left[\frac{b\cos(bw)}{w} - \frac{\sin(bw)}{w^2}\right] = \frac{1}{b}\left[\frac{bw\cos(bw) - \sin(bw)}{w^2}\right]$$

$$\lim_{w\to 0} \frac{d\Phi_X(w)}{dw} = \frac{1}{b}\left[\lim_{w\to 0} \frac{bw\cos(bw) - \sin(bw)}{w^2}\right] = \frac{1}{b}\left[\lim_{w\to 0} \frac{b\cos(bw) - b^2 w\sin(bw) - b\cos(bw)}{2w}\right]$$

$$= \frac{1}{b}\left[\lim_{w\to 0} \frac{-b^2 w \sin(bw)}{2w}\right] = \frac{1}{b}\left[\frac{-b^2\sin(bw)}{2}\right]\Bigg|_{w=0} = 0$$

$$\Rightarrow E[X] = \frac{1}{j}(0) = 0 \rightsquigarrow \boxed{E[X]=0} \leftarrow \text{Mean}$$

$$\frac{d^2\Phi_X(w)}{dw} = \frac{1}{b}\left[\frac{-b^2\sin(bw)}{w} - \frac{b\cos(bw)}{w^2} - \frac{b\cos(bw)}{w^2} + \frac{2\sin(bw)}{w^3}\right] = \frac{1}{b}\left[\frac{-b^2\sin(bw)}{w} - \frac{2b\cos(bw)}{w^2} + \frac{2\sin(bw)}{w^3}\right]$$

8 $\frac{}{1}$ $\frac{}{2}$ $\frac{}{3}$ $\frac{T}{r}$

8. (6+5+4 pts Consider a biased coin with p being the probability of heads. We flip the coin until r tails have appeared, and then stop flipping the coin. Let X be the random variable denoting the number of heads in this experiment.

a) Find the PMF of X — Heads w.p p
(b) Find the expected value of X — Tails w.p $1-p$
c Find the variance of X — Note Last flip MUST be tails

a) $X = \#$ of heads ; $Y =$ total $\#$ of flips so $Y = X + r$

$$P(X=0) = (1-p)^{r}$$

$$P(X=1) = (1-p)^{r} p \binom{r}{1}$$ ← chooses location of one heads (but can't be last spot)

$$P(X=2) = (1-p)^{r} p^{2} \binom{r+1}{2}$$ ← Choose locations of two heads but can't be last spot, hence $\binom{r+2-1}{2}$ options

$$\boxed{P(X=n) = (1-p)^{r} p^{n} \binom{r+n-1}{n}}$$

+6

b) $E[X] = \sum_{i=0}^{\infty} P(X=a_i) \cdot a_i$ +1

c) $Var(X) = E[X^2] - (E[X])^2$ +1