1. (6 pts) Suppose $P(A) = 1/3$, $P(A \cup B) = 2/3$, $P(A \cap B) = 1/4$. Compute $P(B)$.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \quad P(B) = P(A \cup B) - P(A) + P(A \cap B)$$

$$= \frac{2}{3} - \frac{1}{3} + \frac{1}{4}$$

$$= \frac{1}{3} + \frac{1}{4}$$

$$= \frac{7}{12}.$$

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Make your eqn clearer in the future

2. (10 pts) Suppose X is a Binomial RV with parameters $n = 4$, and p. Express $\mathbb{E}[sin(\pi X/2)]$ in terms of p.

$$E[\sin(\tfrac{\pi X}{2})] = \sum_{i=0}^{4} \binom{4}{i} p^{i} (1-p)^{4-i} \cdot \sin(\tfrac{\pi i}{2})$$

	$i =$	0	1	2	3	4
Sine →	$\sin(\frac{\pi i}{2}) =$	0	1	0	-1	0

$$E[\sin(\tfrac{\pi X}{2})] = \binom{4}{1} p(1-p)^3 - \binom{4}{3} p^3(1-p)$$

$$\cancel{= \binom{4}{1}\left(p(1-p)^3 + p^3(1-p)\right)}$$

$$\cancel{= 4p(1-p)^3 + 4p^3(1-p)}$$

$$\cancel{= 4p(1-p)\left[(1-p)^2 + p^2\right]}$$

$$\cancel{= 4p(1-p)\cdot\left[1 + 2p^2 - 2p\right]}$$

$$= 4p(1-p)^3 - 4p^3(1-p)$$

$$= 4p(1-p)\left[(1-p)^2 - p^2\right]$$

$$= 4p(1-p)\left[1-2p\right].$$

$$= 4p(1-p)(1-2p).$$

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3. (4+5+5 pts) Suppose X is a random variable with CDF $F(x)$ given as

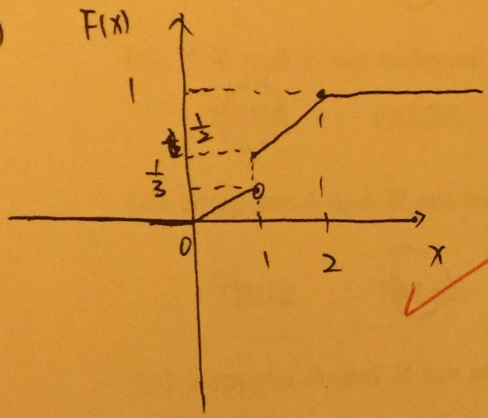
$$F(x) = \begin{cases} 0 & x < 0 \\ x/3 & 0 \le x < 1 \\ x/2 & 1 \le x < 2 \\ 1 & x \ge 2. \end{cases}$$

(a) Sketch F(x)

(b) Compute $P(1/2 \le X \le 3/2)$

(c) Compute $P(1 < X < 2)$

(a)



(b) $P(1/2 \le X \le 3/2) = P(X \le \frac{3}{2}) - P(X < \frac{1}{2})$ ~~$+ P(X = \frac{1}{2})$~~

~~$= P(X \le \frac{3}{2}) - P($~~

~~$= F(\frac{3}{2})$~~

$= F(\frac{3}{2}) - [$~~$F(X \le \frac{1}{2})$~~ $F(\frac{1}{2}^-)]$

$= \frac{3}{4} - \frac{1}{6}$

$= \boxed{\frac{7}{12}}$

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(c) $P(1 < X < 2) = P(X < 2) - P(X \le 1)$

or $F_X(1^+)$

$= F(2^-) - F(1)$

$= 1 - \frac{1}{2}$

$= \boxed{\frac{1}{2}}$

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4. (15 pts) True or False.

Circling the correct answer is worth +3 points, circling the incorrect answer is worth −1 points. Not circling either is worth 0 points.

(a) If X and Y are uncorrelated, they are also independent.

TRUE FALSE ✓

(b) Mean of a random variable is always non-negative.

TRUE FALSE ✓

(c) If X and Y are independent, $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

TRUE FALSE ✓

(d) Suppose A and B are some events. $P(A \cup B) = P(A) + P(B)$.

TRUE FALSE ✓

(e) Suppose A and B are some events. $P(A)P(B|A) = P(B)P(A|B)$.

TRUE FALSE ✓

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5. (8+7 pts)

(a) Suppose a box has 10 balls, labeled 1 through 10. Two balls are selected with replacement, one at the time. Let X denote the larger of the two values. Compute the PMF of X.

(b) Repeat part a), now assuming the experiment is done without replacement.

~~Let A = {total sample space of combinations of~~

(a) There will be ~~$10^2 = 100$ different~~ $10^2 = 100$

different outcomes in total.

Among which there will be

$X =$ 1 2 3 ... n ... 10

# of occurences 1 3 5 ... $(n-1)\cdot 2+1 = 2n-1$ ... 19

$$\Rightarrow P(X) = \frac{2X-1}{100}.$$

(b) In total: $\frac{10!}{2!\cdot(10-2)!} = 90$ different situations.

Among which there will be

$X =$ 1 2 3 ... n ... 10

# of occurences 0 2 4 ... $2(n-1)$ ... 18.

$$\Rightarrow P(X) = \frac{2(n-1)}{90} = \frac{n-1}{45}.$$

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Again, writing should be clearer

consider using a pencil

6. (10 pts) A poker hand has 5 cards drawn from an ordinary deck of 52 cards. What is the probability that the poker hand has exactly two queens ?

Total: $\binom{52}{5} = \frac{52!}{47!\,5!}$ ✓

queen queen n not queen

$(52-5)\cdot(52-6)$

Exactly two queens: $\cancel{\binom{4}{2} = 6}$. $4\cdot 3\cdot(52-4)\cdot\cancel{(52-4)\cdot 1}$

That is correct!

$= 12\cdot 48\cdot 47\cdot 46$

$\Rightarrow P(\text{exactly two queens}) = \frac{12\cdot 48\cdot 47\cdot 46}{\frac{52!}{47!\,5!}}$

$= \cancel{\frac{12\cdot 48\cdot 47!}{52!}}$

$= \frac{12\cdot 48\cdot 47\cdot 46\cdot 5!}{52\cdot 51\cdot 50\cdot 49\cdot 48}$

$= \frac{12\cdot 120\cdot 47\cdot 46}{52\cdot 51\cdot 50\cdot 49}$

$\frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}$?

$= 1440\cdot\frac{47!}{45!}\cdot\frac{48!}{52!}$

$= \frac{1440\cdot 47!\,48!}{52!\,45!}$ ✗

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You are overcounting because your numerator is with ordering while your denom is without ordering

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Ans: $p = \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}$

7. (10 pts) A basketball player has 1/3 chance of scoring a basket. If he makes 8 attempts, what is the probability that he scores at least twice ?

At least twice $\Leftrightarrow$ $\begin{cases} \text{Not none} \\ + \\ \text{Not once} \end{cases}$

$P(\text{scores none}) = \left(1-\frac{1}{3}\right)^8 = \left(\frac{2}{3}\right)^8.$

$P(\text{scores exactly once}) = \binom{8}{1} \cdot \frac{1}{3} \cdot \left(1-\frac{1}{3}\right)^7 = 8 \cdot \frac{1}{3} \cdot \left(\frac{2}{3}\right)^7 = \frac{2^{10}}{3^8}$

$\Rightarrow P(\text{At least twice}) = 1 - P(\text{scores none}) - P(\text{scores exactly one})$

$= 1 - \frac{2^8}{3^8} - \frac{2^{10}}{3^8}$

$= 1 - \frac{2^8(1+2^2)}{3^8}$

$= 1 - \frac{5}{3^8} \cdot 2^8$

$= 1 - 5\left(\frac{2}{3}\right)^8.$

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$\sum_{k=0}^{n-1} \frac{n}{n-k}$ $n \ln n$

8. (20 pts) Suppose we want to collect n different coupons. We get coupons by buying boxes of chocolate: Each box contains exactly one coupon, and it is equally likely that a given box contains any of the n different coupons. What is the expected number of boxes of chocolate we need to buy until we have collected at least one copy of each coupon ? *Hint: Try to decompose the event of interest into a series of simpler events.*

Bonus question: Approximate the expression for large n.

Name ~~Let~~ the different n coupons: $C_1, C_2 \cdots C_n$ for ~~the differen n~~

When buying one box: $P(\text{get } C_i) = \frac{1}{n}$. Let N = # of boxes we need to buy to get all different coupons.

~~So let $X_1, \dots X_n$~~ We want $E[N]$. $N = \{n, n+1 \cdots \infty\}$.

~~Let X_i denote $X_i = 1$ if we get C_i, and $X_i = 0$ if we don't get C_i.~~

→ Let $X_i = k$ if we get C_i the first time when buying k times.

~~We want $E[N]$.~~

Let $p = \frac{1}{n}$. (get C_i) then $1-p \Rightarrow$ (not getting C_i).

For specific C_i. $E[X_i] = \sum_{k=1}^{\infty} k(1-p)^{k-1} p = \frac{1}{p} = n$.

$N = \max$ of X_i. $1 \le i \le n$. $= \frac{n^k}{k!} e^{-n}$.

$E[N] = \sum_{k=n}^{\infty} k \cdot \frac{n^k}{k!} e^{-n}$

$n \to \infty$ $E(N) = \sum_{k=n}^{\infty} k \cdot \frac{n^k}{k!}$

$= \sum_{k=n}^{\infty} \frac{n^k}{(k-1)!}$

metric

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Sum of geometric rvs

$E(C_n^t) = 1 + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + n$

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