EE 131A
Probability
Instructor: Lara Dolecek

Winter 2016 Midterm Version B
Monday, February 1st, 2016

76

Maximum score is 100 points. You have 110 minutes to complete the exam. Please show your work. Good luck!

Your Name:

Your ID Number:

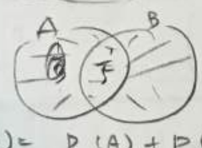
Name of person on your left: ~~Minseong~~ Park

Name of person on your right: Minseon Park

Problem	Score	Possible
1	6	6
2	10	10
3	14	14
4	11	15
5	10	15
6	10	10
7	9	10
8	6	20
Total	76	100

$\frac{3}{4} - \frac{1}{3}$

1. (6 pts) Suppose $P(A) = 1/3$, $P(A \cup B) = 3/4$, $P(A \cap B) = 1/5$. Compute $P(B)$.



$\frac{3}{4}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) - P(A) + P(A \cap B) = P(B)$

$\frac{3}{4} - \frac{1}{3} + \frac{1}{5} = \frac{45}{60} - \frac{20}{60} + \frac{12}{60} = \boxed{\frac{37}{60}}$

45 12   57 20   25 32

6

2. (10 pts) Suppose X is a Binomial RV with parameters $n = 4$ and p. Express $\mathbb{E}[cos(\pi X/2)]$ in terms of p.

discrete $\binom{1}{0} \to (p)$

$P(X=k) = \binom{4}{k} p^k (1-p)^{4-k}$

X Binomial RV n, p $E(X) =$

Sol) $\Rightarrow$ $P(X=k) = \binom{4}{k} p^k (1-p)^{4-k}$; $k \in \{0,1,2,3,4\}$

$Y = \cos\left(\frac{\pi X}{2}\right)$

$E(X) = \sum X_i P_i$, $E(Y) = \sum Y_i P_i$ and $k \in \{0,1,2,3,4\}$

$P(X=0) = (1-p)^4$ and let $1-p = q$

probability

$$\Rightarrow \begin{cases} P(X=0) = q^4 \Leftrightarrow P(Y=0) & X=0 \to Y=1, & q^4 \\ P(X=1) = 4pq^3 \Leftrightarrow P(Y=\cos\frac{\pi}{2}) & X=1 \to Y=0, & 4pq^3 \\ P(X=2) = 6p^2q^2 \Leftrightarrow P(Y=\cos\pi) & X=2 \to Y=-1, & 6p^2q^2 \\ P(X=3) = 4p^3q \Leftrightarrow P(Y=\cos\frac{3\pi}{2}) & X=3 \to Y=0, & 4p^3q \\ P(X=4) = p^4 \Leftrightarrow P(Y=\cos 2\pi) & X=4 \to Y=1, & p^4 \end{cases}$$

now we need to find $E[Y]$

$E[Y] = \sum Y_i P_i = 1 \cdot q^4 + 0 \cdot 4pq^3 + (-1) \cdot 6p^2q^2 + 0 \cdot 4p^3q + 1 \cdot p^4$

$$\boxed{E[Y] = (1-p)^4 - 6p^2(1-p)^2 + p^4}$$

10

3. (4+5+5 pts) Suppose X is a random variable with CDF $F(x)$ given as

$$F(x) = \begin{cases} 0 & x < 0 \\ x/4 & 0 \le x < 2 \\ x/3 & 2 \le x < 3 \\ 1 & x \ge 3. \end{cases}$$

(a) Sketch F(x)

(b) Compute $P(3/2 \le X \le 5/2)$

(c) Compute $P(2 < X < 3)$

$F(a)$ $(\; < x < \;)$ a.

$F(a) = p(x \le a)$

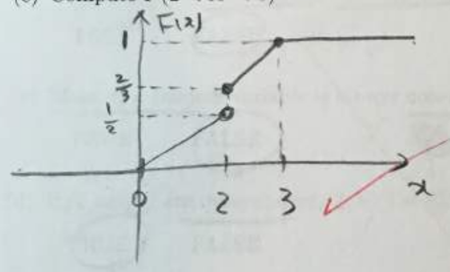
$\underline{F(b) - F(a)}$ $a < x \le b)$

$F(b) - F(a) = p(a < x \le b)$

$\frac{5}{6} - \frac{3}{8}$

$\frac{20}{24} - \frac{9}{24}$

1(a)



(b) $p(3/2 \le X \le 5/2)$

$p(3/2 \le x \le 5/2) = F(\underset{2.5}{5/2}) - F(\underset{1.5}{3/2})$

$F(5/2) = \frac{5}{2} \cdot \frac{1}{3}$, $F(3/2) = \frac{3}{2} \cdot \frac{1}{4}$

$p(3/2 \le x \le 5/2) = \frac{5}{6} - \frac{3}{8} = \frac{20-9}{24} = \boxed{\frac{11}{24}}$

(c) $p(2 < X < 3)$

$F(3) - F(2^+) = 1 - \frac{2}{3} = \boxed{\frac{1}{3}}$

good

14

4. (15 pts) True or False.

Circling the correct answer is worth +3 points, circling the incorrect answer is worth −1 points. Not circling either is worth 0 points.

(a) Suppose A and B are some events. $P(A)P(B|A) = P(B)P(A|B)$.

TRUE FALSE X $p(A)\,p(B|A) = p(A)\,\cancel{p}(B \cap A)\,p(A)$

-1 $p(B)\,p(A|B) = p(\cancel{A})\,p(A \cap B)\,p(B)$

(b) If X and Y are uncorrelated, they are also independent.

TRUE FALSE ✓

(c) Mean of a random variable is always non-negative.

TRUE FALSE ✓ $E(X)$

(d) If X and Y are independent, $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

TRUE FALSE ✓

(e) Suppose A and B are some events. $P(A \cup B) = P(A) + P(B)$.

TRUE FALSE ✓ $p(A \cup B) \leq p(A) + p(B)$

11

10 balls

1 2 3 4 5 6 7 8 9 $\quad$ 1 —— 10 $\quad$ 00 $\quad \binom{10+2-1}{2} \quad \frac{11}{2} \quad \frac{5 \cdot 10 \cdot 11}{2}$

45

5. (8+7 pts)

(a) Suppose a box has 10 balls, labeled 1 through 10. Two balls are selected without replacement, one at the time. Let X denote the larger of the two values. Compute the PMF of X.

larger value

(b) Repeat part a), now assuming the experiment is done with replacement.

1 2 3 4 5 6 7 8 9 10

number of all outcomes $= \binom{10}{2} = \frac{5 \cdot 10 \cdot 9}{2 \cdot 1} = 45$ $\qquad (1,2) \to X=2$

$(3,7) \to X=7$

(a)

$S_X = \{2,3,4,5,6,7,8,9,10\}$

$P(X=2) = \frac{1}{45}$ $\qquad X=2 \Rightarrow \{(1,2)\}$

$P(X=3) = \frac{2}{45}$ $\qquad X=3 \Rightarrow \{(1,3),(2,3)\}$

$P(X=4) = \frac{3}{45}$ $\qquad \{(1,4)(2,4)(3,4)\}$

$P(X=5) = \frac{4}{45}$ $\qquad \{(1,5)(2,5)(3,5)(4,5)\}$

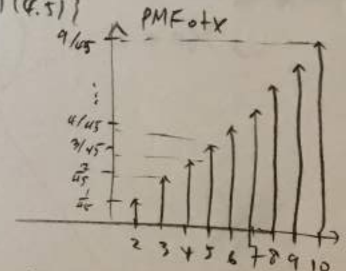
$P(X=6) = \frac{5}{45}$

$P(X=7) = \frac{6}{45}$

$P(X=8) = \frac{7}{45}$

$P(X=9) = \frac{8}{45}$

$P(X=10) = \frac{9}{45}$ $\qquad X=10 \Rightarrow \{(1,10)(2,10) \cdots (9,10)\}$



PMF of X

$\boxed{P(X=k) = \frac{k-1}{45} \; ; \; k = 1,2,3,\cdots,10}$

10

(b) $S_X = \{1,2,3,4,5,6,7,8,9,10\}$ → with repl w/o ordering

total number of outcomes $= 45 + 10 = 55 = \binom{10+2-1}{2}$ X $= (10)^2 = 100$

$P(X=1) = \frac{1}{55}$ $\quad \cdots \quad P(X=10) = \frac{10}{55}$

$P(X=2) = \frac{2}{55}$

$P(X=3) = \frac{3}{55}$ $\qquad \Rightarrow \boxed{P(X=k) = \frac{k}{55} \; ; \; k = 1,2,\cdots,10}$ X

$\frac{2k-1}{100}$

denominator = 100

6

52 ⊕ 5 $\frac{13 \times 4}{2}$

6. (10 pts) A poker hand has 5 cards drawn from an ordinary deck of 52 cards. What is the probability that the ~~poker hand~~ has exactly three queens ?

S ♤ A 2 3 4 5 6 7 8 9 10 J Q K. 13

D ◇ Q 13 52−3

H ♡ Q 13

C ♧ 13 49 48

all outcomes

Sol) pick 5 among 52 = $\binom{52}{5} = {}_{52}C_5$

(SQ, DQ, HQ ‖ _ _)

Queen. not Queen

$\binom{4}{3} = \binom{4}{1}$ ⇒ num of picking three queens , $\binom{48}{2}$ ⇒ number of picking 2 cards among rest of deck not including queen

number of hands pick 3 Queens exactly = $4 \times \binom{48}{2} = 4 \times {}_{48}C_2$

$$\boxed{\text{probability} = \frac{4 \times {}_{48}C_2}{{}_{52}C_5}}$$

10

7. (10 pts) A basketball player has 2/3 chance of scoring a basket. If he makes 8 attempts, what is the probability that he scores at least twice?

$p = \frac{2}{3}$ $\quad n = 8$

geometry

Binomial

$\Rightarrow \quad P(X=k) = (1-p)^{k-1} p$; since geometry ✗

probability he score at least twice

$= 1 -$ probability he fail every 8 attempts

$-$ probability he score only one among 8 attempts

probability of all fail $\Rightarrow \left(\frac{1}{3}\right)^8$

probability of one succeed $\Rightarrow \boxed{\binom{8}{1} \cdot \left(\frac{1}{3}\right)^7 \frac{2}{3}} = \frac{16}{3^8}$

$\boxed{\therefore \text{ probability he scores at least twice} = 1 - \frac{17}{3^8}}$

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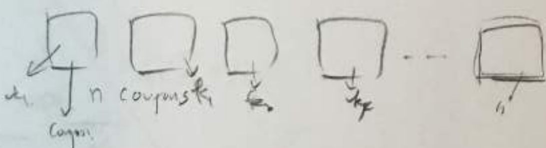
It's binomial & that's what you applied

$n \cdot 4$ Cards from each family?

8. (20 pts) Suppose we want to collect n different coupons. We get coupons by buying boxes of chocolate. Each box contains exactly one coupon, and it is equally likely that a given box contains any of the n different coupons. What is the expected number of boxes of chocolate we need to buy until we have collected at least one copy of each coupon? *Hint: Try to decompose the event of interest into a series of simpler events.*

Bonus question: Approximate the expression for large n.

$(C_1, C_2, \dots C_n)$



probability that

all independent

we get C_1 coupon for first buying $\Rightarrow \frac{1}{n} = p$

$C_2 \Rightarrow \frac{1}{n}$

$\vdots$

$C_n \Rightarrow \frac{1}{n}$

(6)

Expected numbers of boxes until we have C_1 coupon $\Rightarrow n \times \frac{1}{}$

" $C_2 \Rightarrow n \times \frac{n}{n-1}$

ause, $P(X_i = k) = \left(\frac{n-1}{n}\right)^{k-1}\left(\frac{1}{n}\right)$ prob. getting C_i coupon $\rightarrow$ this is probability we get C_i coupon in k trial

and this is geometric RV

$E(X_i) = \sum_{i=1}^{\infty} x_i p_i = \frac{1}{p}, \ p = \frac{1}{n} \ \therefore E(X) = n$

Let $Y = X_1 + X_2 + \dots X_n$; which means probability of collecting all C_i's coupon.

$$\boxed{E(Y) = E(X_1 + X_2 + \dots X_n) = E(X_1) + E(X_2) + \dots + E(X_n) = n^2}$$ X

Bonus: If we have 100 different Coupon, then Expectation numbers of buying box of chocolate to collecting at least one copy of each coupon is $100^2 = 10000$

Sum of geometric rvs

$$E(C_n^t) = 1 + \frac{n}{n-1} + \frac{n}{n-2} + \dots + n$$

EE 110, Winter 2016, Midterm Exam – February 3, 2016

Instructions: This exam booklet consists of three problems, blank sheets for the solutions, reference sheets with mathematical identities, and additional blank sheets. Please follow these instructions while answering your exam:

1. **Write your name and student identification number below.**
2. **Write the names of students to your left and right as well.**
3. **You have 1 hour 45 minutes to finish your exam.**
4. **Write your solutions in the provided blank sheets after each problem.**
5. **The sheets marked "Scratch…" will NOT be graded. These sheets are provided for your rough calculations only.**
6. **Write your solutions clearly. Illegible solutions will NOT be graded.**
7. **Be brief.**

NAME: [illegible]

STUDENT ID: ____________

NAMES OF ADJACENT STUDENTS:

LEFT: Juan Morales

RIGHT: Edison Vallejo