Midterm solutions   Ahmed Hareedy   Version A

1- Suppose $P(A) = 1/3$, $P(A \cup B) = 2/3$, $P(A \cap B) = 1/4$.
Compute $P(B)$.

Sol- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow P(B) = P(A \cup B) - P(A) + P(A \cap B)$

$\Rightarrow P(B) = \frac{2}{3} - \frac{1}{3} + \frac{1}{4} = \frac{1}{3} + \frac{1}{4} = \boxed{\frac{7}{12}}$

2- Suppose X is a Binomial RV with parameters $n=4$, and p. Express $E\left[\sin\left(\frac{\pi X}{2}\right)\right]$ in terms of p.

Sol. X Binomial $\Rightarrow P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$, $n=4$

$\Rightarrow E\left[\sin\left(\frac{\pi X}{2}\right)\right] = \sum_{x=0}^{4} \sin\left(\frac{\pi x}{2}\right) \binom{4}{x} p^x (1-p)^{4-x}$

$= \cancel{\sin(0)\binom{4}{0} p^0 (1-p)^4} + \sin\left(\frac{\pi}{2}\right)\binom{4}{1} p (1-p)^3 + \cancel{\sin(\pi)\binom{4}{2} p^2 (1-p)^2}$

$+ \sin\left(\frac{3\pi}{2}\right)\binom{4}{3} p^3 (1-p) + \cancel{\sin(2\pi)\binom{4}{4} p^4 (1-p)^0}$

$= 4p(1-p)^3 - 4p^3(1-p) = \boxed{4p(1-p)(1-2p)}$

3- Suppose X is a random variable with CDF $F_X(x)$ given as

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x/3, & 0 \le x < 1 \\ x/2, & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$

(a) Sketch $F_X(x)$

(b) Compute $P(1/2 \leq X \leq 3/2)$

Sol. $P(1/2 \leq X \leq 3/2)$

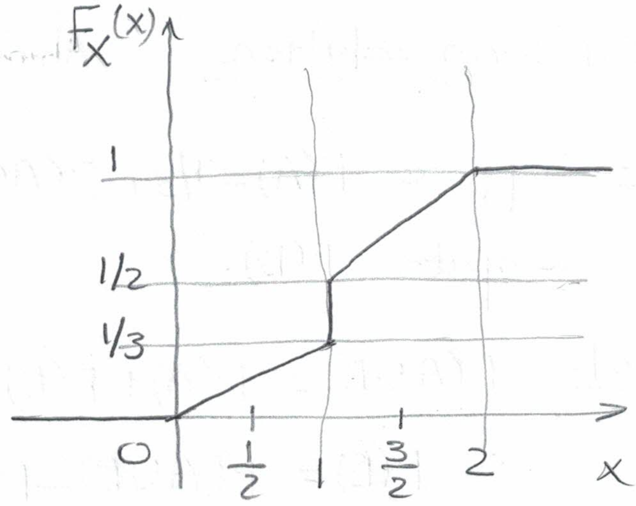
$= F_X\left(\frac{3}{2}\right) - F_X\left(\frac{1}{2}\right)$

$= \frac{3/2}{2} - \frac{1/2}{3} = \frac{3}{4} - \frac{1}{6} = \boxed{\frac{7}{12}}$

(c) Compute $P(1 < X < 2)$

Sol. $P(1 < X < 2) = F_X(2) - F_X(1^+)$

$= 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$



Sol. (a)

4. (a) If X and Y are uncorrelated, they are also independent.

Sol. False

(b) Mean of a random variable is always non-negative.

Sol. False

(c) If X and Y are independent, $E[XY] = E[X]E[Y]$.

Sol. True

(d) Suppose A and B are some events. $P(A \cup B) = P(A) + P(B)$.

Sol. False

(e) Suppose A and B are some events.

$P(A)P(B|A) = P(B)P(A|B)$

Sol. True

5. Suppose a box has 10 balls, labeled 1 through 10. Two balls are selected with replacement, one at the time. Let X denote the larger of the two values. Compute the PMF of X. (a)

Sol. Total number of outcomes $= 10^2 = 100$

$$P_X(x) = P(X=x) = \boxed{\frac{2x-1}{100}}, \quad x \in \{1, 2, \cdots, 10\}$$

(b) Repeat part (a), now assuming the experiment is done without replacement.

Sol. Total number of outcomes $= (10)(9) = 90$

$$P_X(x) = P(X=x) = \boxed{\frac{2x-2}{90}}, \quad x \in \{1, 2, \cdots, 10\}$$

Note $P(X=1) = \frac{2-2}{90} = 0$ for part (b).

6. A poker hand has 5 cards drawn from an ordinary deck of 52 cards. What is the probability that the poker hand has exactly two queens?

Sol. Ways of selecting 2 queens out of $4 = \binom{4}{2}$

Ways of selecting 3 cards (not queens) $= \binom{48}{3}$

Ways of selecting 5 cards out of $52 = \binom{52}{5}$

$$P(2 \text{ queens in hand}) = \boxed{\frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}}$$

7. A basketball player has 1/3 chance of scoring a basket. If he makes 8 attempts, what is the probability that he scores at least twice?

Sol. $p(X \geq 2) = 1 - p(X < 2) = 1 - p(X=1) - p(X=0)$

$$= 1 - \binom{8}{1} p^1 (1-p)^7 - \binom{8}{0} p^0 (1-p)^8$$

$$= 1 - 8\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^7 - (1)\left(\frac{2}{3}\right)^8$$

$$= 1 - \left(\frac{2}{3}\right)^7\left(\frac{8}{3} + \frac{2}{3}\right) = \boxed{1 - \left(\frac{10}{3}\right)\left(\frac{2}{3}\right)^7}$$

8. Collecting coupons:

$E(\text{total boxes to collect } n) = E(C_n^t)$

$$= E(C_1 + C_2 + \cdots + C_n),$$

where C_i is a geometric RV that represents the number of trials (boxes) needed to collect the i^{th} coupon.

C_1 has $P_1 = 1 \quad \Rightarrow E(C_1) = \frac{1}{P_1} = 1$

C_2 has $P_2 = \frac{n-1}{n} \quad \Rightarrow E(C_2) = \frac{1}{P_2} = \frac{n}{n-1}$

C_3 has $E(C_3) = \frac{n}{n-2}$

… until C_n that has $E(C_n) = n$

$$E(C_n^t) = E\left(\sum_{i=1}^{n} C_i\right) = \sum_{i=1}^{n} E(C_i) = 1 + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + n$$

$$= n\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) = \boxed{n \sum_{k=1}^{n} \frac{1}{k}}$$

Bonus $E(C_n^t) = n H_n = n\left(\ln n + \gamma + \frac{O(1)}{n}\right) = n \ln n + \gamma n + O(1)$

For large $n \rightarrow E(C_n^t) \approx \boxed{n \ln n + O(n)}$, H_n: n^{th} Harmonic number