Midterm Solutions

EE 131A
Probability
Instructor: Lara Dolecek

Fall 2010 Midterm
Wednesday, October 27, 2010

Maximum score is 100 points. You have 110 minutes to complete the exam. Please show your work. Good luck!

Your Name:

Your ID Number:

Name of person on your left:

Name of person on your right:

Problem	Score	Possible
1		6
2		15
3		8
4		12
5		15
6		8
7		8
8		10
9		18
Total		100

1. (6 pts) Suppose $P(A) = 1/3$, $P(A \cup B) = 1/2$ and $P(A \cap B) = 1/5$. Find $P(B)$.

Sol: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(B) = P(A \cup B) - P(A) + P(A \cap B) = \frac{1}{2} - \frac{1}{3} + \frac{1}{5} = \frac{11}{30}$

2. (4+5+6 pts) Let X and Y be independent and uniform on $\{1, 2, \dots M\}$. Find

(a) $P(X = Y)$.

(b) $P(X \geq Y)$.

(c) pmf of U where $U = |X - Y|$.

Sol. (a) $P(X=Y) = \frac{M}{M^2} = \frac{1}{M}$

(b) $P(X \geqslant Y) = \frac{(M+1)M/2}{M^2} = \frac{M+1}{2M}$

(c) $P(U=u) = \begin{cases} \frac{1}{M} & \text{if } u=0 \\ \frac{2(M-u)}{M^2} & u=1,2,3\cdots M-1 \end{cases}$

3. (4+4 pts) Suppose X and Y are independent random variables with finite first and second moments. Let $Z = 3X + 5Y$. Compute mean and variance of Z in terms of $E(X), E(Y), VAR(X)$ and $VAR(Y)$.

Sol: $E(Z) = E(3X+5Y) = 3E(X) + 5E(Y)$

$Var(Z) = Var(3X+5Y) = 9\,Var(X) + 25\,Var(Y)$

4. (12 pts) A 52-card deck consists of 4 suits (clubs, diamonds, hearts, spades), and each suit has 13 cards (2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A). Suppose you are dealt a poker hand consisting of 5 cards. Compute the probability of getting flush. Flush hand is 5 cards of the same suit.

Sol: $p(\text{flush}) = \dfrac{4\binom{13}{5}}{\binom{52}{5}} \approx 0.20\%$

5. (15 pts) True or False.

Circling the correct answer is worth +3 points, circling the incorrect answer is worth −1 points. Not circling either is worth 0 points.

(a) Suppose $X \sim \mathcal{N}(0,1)$. Then $P(X = 0) = 0.5$.

TRUE   FALSE

(b) If A and B are mutually exclusive events, $P(A \cup B) = P(A) + P(B) - P(A)P(B)$.

TRUE   FALSE

(c) $E[(X - m)^2] = E[X^2] - m^2$ where $m = E[X]$.

TRUE   FALSE

(d) $VAR(X + c) = VAR(X) + c$ (where c is a constant).

TRUE   FALSE

(e) $VAR(X - Y) = VAR(X) - VAR(Y)$

TRUE   FALSE

6. (8 pts) Suppose that in a certain city 60% of residents are male and 40% of residents are female. Suppose also that 50% of males and 30% of females cheer for Lakers. Find the probability that a Lakers fan is female.

Sol. $P(\text{female} \mid \text{Lakers fan}) = \dfrac{P(\text{female Lakers fan})}{P(\text{Lakers fan})}$

$$= \frac{0.4 \cdot 0.3}{0.4 \cdot 0.3 + 0.6 \cdot 0.5}$$

$$= \frac{2}{7}$$

7. (4+4 pts) Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$.

(a) Find $P(|X - \mu| \leq \sigma)$ in terms of Q function.

(b) Find constants a and b such that $Z = a + bX$ is $\mathcal{N}(0, 1)$.

Sol: (a) $P(|X-\mu| \leq \sigma) = 1 - 2P(X-\mu > \sigma)$

$= 1 - 2Q(1)$

(b) $Z = \frac{X-\mu}{\sigma} = -\frac{\mu}{\sigma} + \frac{1}{\sigma}X$ is $N(0,1)$

$a = -\frac{\mu}{\sigma}$ $\quad b = \frac{1}{\sigma}$

8. (8+2 pts) Let X be uniformly distributed on $(0,1)$. Find cdf and pdf of $Y = -\frac{1}{a}\ln(1-X)$, where $a > 0$. What is the name of this random variable ?

Sol. $P(Y \le y) = P[-\frac{1}{a}\ln(1-X) \le y] = P[\ln(1-X) \ge -ay]$

$= P[1-X \ge e^{-ay}] = P[X \le 1-e^{-ay}] = 1-e^{-ay}$ for $y>0$.

$F_Y(y) = 1-e^{-ay}$ for $y>0$

$f_Y(y) = \begin{cases} 0 & \text{otherwise} \\ a e^{-ay} & \text{if } y>0 \end{cases}$

exponential random variable

9. (4+6+8 pts) Suppose n balls are distributed in r boxes. Let $X_i = 1$ if box i is empty, and 0 otherwise, for $1 \le i \le r$.

(a) Compute $E(X_i)$ for a fixed i, $1 \le i \le r$.

(b) Compute $E(X_i X_j)$ for fixed i, j, where $1 \le i, j, \le r$ and $i \ne j$.

(c) Let S denote the number of empty boxes. Compute mean and variance of S.

Sol: (a) $E(X_i) = \sum_{X_i} P(X_i) \cdot X_i = P(X_i = 1) = \frac{(r-1)^n}{r^n}$

(b) $E(X_i X_j) = P(X_i = 1 \text{ and } X_j = 1) = \frac{(r-2)^n}{r^n}$

(c) $S = \sum_{i=1}^{r} X_i$

$E(S) = E(\sum_{i=1}^{r} X_i) = \sum_{i=1}^{r} E(X_i) = r \frac{(r-1)^n}{r^n} = \frac{(r-1)^n}{r^{n-1}}$

$E(S^2) = E[(\sum_{i=1}^{r} X_i)^2] = r E(X_i^2) + r(r-1) E(X_i X_j)$

$E(X_i^2) = \sum P(X_i) X_i^2 = P(X_i = 1) = \frac{(r-1)^n}{r^n}$

$Var(S) = E(S^2) - E(S)^2 = r \cdot \frac{(r-1)^n}{r^n} + r(r-1) \frac{(r-2)^n}{r^n} - \frac{(r-1)^{2n}}{r^{2n-2}}$

$= \frac{(r-1)^n}{r^{n-1}} + \frac{(r-1)(r-2)^n}{r^{n-1}} - \frac{(r-1)^{2n}}{r^{2n-2}}$