EE 131A
Probability
Instructor: Lara Dolecek

Winter 2016 Final
Monday, March 14th, 2016

**Maximum score is 100 points. You have 180 minutes to complete the exam. Please show your work.**

**Good luck!**

# Your Name:

# Your ID Number:

# Name of person on your left:

# Name of person on your right:

| Problem | Score | Possible |
|---|---|---|
| 1 | | 12 |
| 2 | | 12 |
| 3 | | 15 |
| 4 | | 11 |
| 5 | | 16 |
| 6 | | 11 |
| 7 | | 18 |
| 8 | | 5 |
| Total | | 100 |

1. (12 pts) Suppose $X$ is a uniform random variable in the interval $[0, \pi/2]$.

(a) (2 pts) Find $P[X \geq \pi/4]$ and $P[X \geq 3\pi/4]$.

(b) (2 pts) Use Markov inequality to find an upper bound on $P[X \geq \pi/4]$, and also an upper bound on $P[X \geq 3\pi/4]$.

(c) (2 pts) Compare your results of (a) and (b). Why is the bound loose?

(d) (6 pts) Now suppose $X_1, X_2, \ldots, X_n$ are all independent, identically distributed uniform random variables in the interval $[0, \pi/2]$, find (approximately):

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\sin(X_i)\cos(X_i)\right).$$

(a) $P[X \geq \frac{\pi}{4}] = 1 - P[X < \frac{\pi}{4}]$

$= 1 - P[X \leq \frac{\pi}{4}]$

$= 1 - \int_0^{\frac{\pi}{4}} \frac{2}{\pi}\, dx$

$= 1 - \frac{\pi}{4} \times \frac{2}{\pi}$

$= \frac{1}{2}$

$P[X \geq \frac{3}{4}\pi] = 1 - P[X \leq \frac{3}{4}\pi] = 1 - 1 = 0$.

(b) $E[X] = \frac{1}{2}[\frac{\pi}{2} + 0] = \frac{\pi}{4}$. $\quad P[X \geq \frac{\pi}{4}] \leq \frac{\frac{\pi}{4}}{\frac{\pi}{4}} = 1$

$P[X \geq \frac{3}{4}\pi] = \frac{\frac{\pi}{4}}{\frac{3}{4}\pi} = \frac{1}{3}$

(c) Since Markov inequality only consider first Moment of a RV.

(d) Let $M_n = \frac{1}{n}\sum_{i=1}^{n} Y_i \qquad Y_i = \sin(X_i)\cos(X_i)$

$E[M_n] = E[Y_i] = E[\sin(X_i)\cos(X_i)]$

$= \int_0^{\frac{\pi}{2}} \sin x \cos x \frac{2}{\pi}\, dx$

$= \frac{1}{\pi}.$

From LLN, $\lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n}\sin(X_i)\cos(X_i)\right] = E[M_n] = \frac{1}{\pi}$

2. (12 pts) Carlos and Michael are participating in a tennis tournament that has 16 players including them. The tournament ladder is shown below. The winner of any match advances to the next round while the loser is eliminated. Assume that the initial draw is random and unbiased. Assume also that all the 16 players are evenly matched (i.e., when any two of them meet, either one can win the probability 1/2).

   (a) (6 pts) Compute the probability that Carlos and Michael meet in the tournament.

   (b) (6 pts) Repeat part (a) if the number of players is $2^n$ rather than 16.

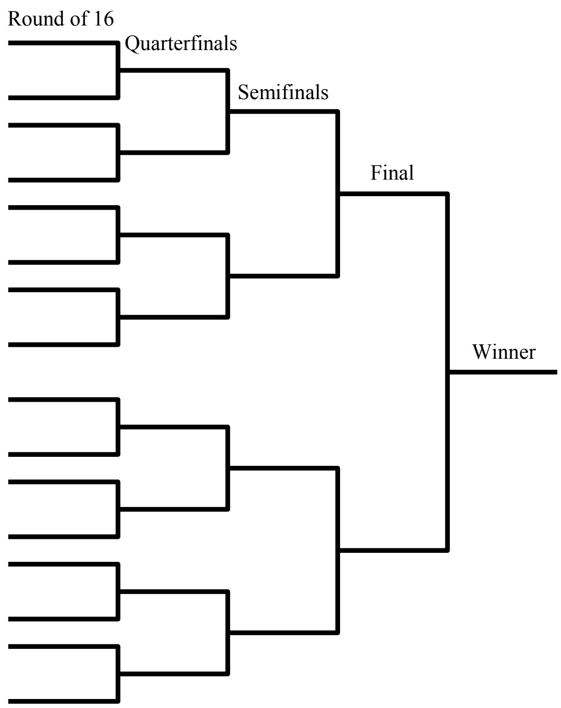


Figure 1: The ladder of the tennis tournament of problem 2.

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a) ① $p(\text{meet in } 1^{st} \text{ round}) = \frac{1}{15}$ probability of assign position.

② $p(\text{meet in } 2^{nd} \text{ round}) = \frac{2}{15}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{15}\left(\frac{1}{2}\right)$ both win.

③ $p(\text{meet in } 3^{rd} \text{ round}) = \frac{4}{15}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = \frac{1}{15}\left(\frac{1}{2}\right)^2$

④ $p(\text{meet in } 4^{th} \text{ round}) = \frac{8}{15}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^3 = \frac{1}{15}\left(\frac{1}{2}\right)^3$.

$p(\text{meeting}) = ① + ② + ③ + ④ = \frac{1}{8}$.

b) $p(\text{meeting}) = \frac{1}{2^n-1}\left(1+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2+\cdots+\left(\frac{1}{2}\right)^{n-1}\right)$

$= \frac{1}{2^n-1}\sum_{i=0}^{n-1}\left(\frac{1}{2}\right)^i = \frac{1}{2^n-1}\,\frac{(1)\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\left(\frac{1}{2}\right)}$

$= \frac{1}{2^n-1}\left(\frac{2^n-1}{2^n}\right)(2) = \frac{1}{2^{n-1}}$

a) $\phi_Y(w) = E[e^{jwz}] = E[e^{jw(aX+b)}]$

$= e^{jwb} E[e^{jwaX}]$

3. (15 pts) True or False. $= e^{jwb} \Phi_X(aw)$

**Circling the correct answer is worth +3 points, circling the incorrect answer is worth −1 points**. Not circling either is worth 0 points.

(a) Suppose $X$ is a continuous random variable, and $Y = aX + b$. The characterisitic function of $Y$ can be obtained from the characterisitc function of $X$ using:

$$\Phi_Y(\omega) = e^{j\omega b}\Phi_X(a\omega).$$

TRUE FALSE

(b) Let $X$ be a random variable and $g(X)$ is a function of $X$, then for any $X$ and any $g(.)$, $E[g(X)|X] = g(X)$.

TRUE FALSE $E[E[g(X)|X]] = E[g(X)] \not\Rightarrow$

$E[g(X)|X] = g(X)$

(c) If $X$ is a random variable and $Y = aX$, $-\infty < a < \infty$, then $\rho_{X,Y} = 1$ ($\rho_{X,Y}$ is the correlation coefficient betwen $X$ and $Y$).

TRUE FALSE $a \neq 0$. if $a = 0$ $\rho_{X,Y} \neq 1$

(d) The probability generating function $G_X(z)$ of a random variable $X$, is a function defined over discrete values of $z$. PGF is for discrete RV.

TRUE FALSE

but $z$ is not discrete.

(e) Suppose $X_1, X_2, \ldots, X_n$ are dependent geometric random variables, and $S_n = X_1 + X_2 + \cdots + X_n$. Using the central limit theorem, $S_n$ tends to be a Gaussian random variable as $n$ tends to $\infty$.

TRUE FALSE

4. (11 pts) Suppose $X$ and $Y$ are two independent Gaussian random variables, each with mean zero and variance $\sigma^2$. Let $Z = \sqrt{X^2 + Y^2}$.

   (a) (8 pts) Find the CDF $F_Z(z)$ and the PDF $f_Z(z)$ of the random variable $Z$.

   (b) (3 pts) Prove that $f_Z(z)$ is indeed a valid PDF.
   *Hint: If you cannot find $f_Z(z)$ of part (a), write down the procedure you should follow to prove whether a function is a valid PDF.*

HW #8 problem 4.

5. (16 pts) For the following problem, ignore the year of birth while comparing two birthdays. Moreover, assume that the year is exactly 365 days (ignore the 29th of February). Note that matching birthdays means the birthdays are the same.

   (a) (4 pts) You are a member of a class room that has $n+1$ students including you. What is the probability that you find at least one student, other than you, who has a birthday that matches yours?

   (b) (5 pts) In another classroom that has $m$ students, what is the probability that at least two students have matching birthdays?

   (c) (7 pts) Now let $P_1$ be the probability that you do not find any student, in your classroom that has $n+1$ students, who has a birthday that matches yours. Additionally, let $P_2$ be the probability that there are no birthday matches in the room that has $m$ students. Derive the relation between $n$ and $m$ such that $P_1 = P_2^2$.
   *Hint: The following approximation will be useful: for small $x$, $1 - x \approx e^{-x}$.*

Hw #2 problem 4.

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6. (11 pts) Consider the jointly Gaussian random variables $X$ and $Y$ that have the following joint PDF:

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} + \frac{y^2}{\sigma_Y^2} - \frac{2\rho xy}{\sigma_X\sigma_Y}\right)\right].$$

(a) (5 pts) Derive the marginal PDF of $Y$, $f_Y(y)$.

(b) (6 pts) Prove that $f_{X|Y}(x|y)$ corresponds to another Gaussian random variable, then find its mean and variance.

HW # 8 problem 3.

7. (18 pts) Consider the communication system shown below. The transmitter transmits $X$ that can take one of two values, either 1 with probability $p$ or $-1$ with probability $1-p$, over a noisy channel. The receiver observes $Y = X + N$, and based on $Y$, it decides the value of $Z$ (an estimate of $X$), which can only be either 1 or $-1$. Suppose that the random variable $N$, which represents the noise in this communication system, follows a Laplacian distribution that has the PDF:

$$f_N(n) = \frac{\alpha}{2} e^{-\alpha|n|}, \; -\infty < n < \infty, \, \alpha > 0.$$

(a) (5 pts) Find the expected value and the variance of $N$.

(b) (6 pts) Suppose that the receiver decides $Z = 1$ if $Y = y > T$, and $Z = -1$ if $Y = y \leq T$. Assuming that $-1 < T < 1$, find the decision threshold $T$ as a function of $\alpha$ and $p$.

(c) (5 pts) Define the probability of error in this system to be $P[Z \neq X]$. If $p = 0.5$, what is the probability of error?
*Hint 1: If you cannot find explicitly the decision threshold $T$ of part (b), for this part try to intuitively find $T$ for $p = 0.5$. If you cannot do that, assume $T$ is given when you solve this part.*

(d) (2 pts) Explain the effect of $\alpha$ on the probability of error.
*Hint 2: You can answer this part even without reaching the result of part (c).*

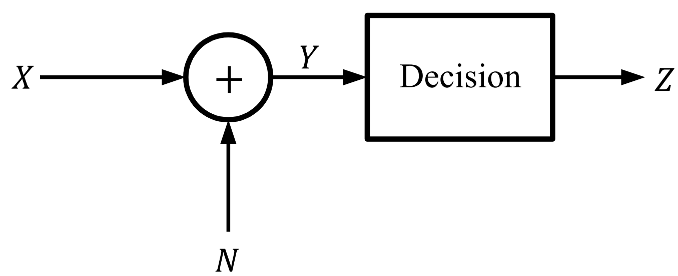


Figure 2: The communication system of problem 7.

HW # 5 pb 6.

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8. (5 pts) Find the expected number of fair coin tosses needed to get two successive tails.
*Hint 1: Do not attempt to reach the underlying probability distribution.*

**Bonus** (10 pts) Repeat the above question for a sequence of $n$ (rather than two) successive tails.
*Hint 2: The following relation will be useful:*

$$\sum_{i=1}^{n} i a^{i-1} = \left[ \frac{d}{dx} \sum_{i=1}^{n} x^i \right]_{x=a}.$$

This is a topic called recursion, you don't need it for the final exam.

Let X be the expected number of coin flips required for getting two tails. in a row.

a. If the first flip turns out to be head, need X more flips since event are iid. Probability of the event $\frac{1}{2}$, Since 1 flip was wasted total number of flips required $(1+X)$.

b. if the first flip becomes tail, but second one is head (TH), 2 flips are wasted. Total number of flips required would be $(2+X)$. probability of TH is $\frac{1}{4}$.

c. Best case, the first two flips turn out to be TT. $p(TT) = \frac{1}{4}$. # of flips is 2.

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$\therefore$ From a, b, c we have the following:

$$X = \frac{1}{2}(1+X) + \frac{1}{4}(2+X) + \frac{1}{4} \times 2$$

$$\Rightarrow X = 6.$$

So the expected number would be 6.