EE 131A
Probability
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Fall 2010 Final
Friday, December 10, 2010

Maximum score is 200 points. You have 180 minutes to complete the exam. Please show your work. Good luck!

Your Name:

Your ID Number:

Name of person on your left:

Name of person on your right:

Problem	Score	Possible
1		15
2		15
3		35
4		10
5		25
6		25
7		15
8		20
9		20
10		20
Total		200

1. (15 pts) You select two numbers (without replacement) from the set $\{1, 2, 3, 4, 5\}$. What is the probability that the sum is at most 6 given that the smaller number is at most 3 ?

Solution. possible outcomes.

(1, 2), (1, 3), (1, 4), (1, 5), (2, 3)

(2, 4) (2, 5) (3, 4), (4, 5) (3, 5)

$P[\text{smaller is at most } 3] = \frac{9}{10}$

$P[\text{smaller} \le 3 \text{ and sum} \le 6] = \frac{6}{10}$

$P[\text{sum} \le 6 \mid \text{smaller} \le 3] = \frac{6}{9} = \frac{2}{3}$

2. (10+5 pts) Suppose X is Gaussian with mean m and variance σ^2.

(a) Find $E[(X-m)^3]$.

(b) Find pdf of $Y = e^X$. Random variable Y is called log-normal.

Solution. (a) Since X is Gaussian with mean m

then $X-m$ is Gaussian with mean 0

the pdf of $X-m$ is symmetric of 0.

its odd moments are 0

$E[(X-m)^3] = 0$

(b) $F_Y(y) = P(Y<y) = P(e^X<y) = P(X<\ln y) = F_X(\ln y)$

$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{1}{y} f_X(\ln y) = \frac{1}{y}\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln y - m)^2}{2\sigma^2}} \quad (y>0)$

3. (10+5+10+10 pts) Suppose X is Poisson r.v. with parameter λ. That is, pmf of X is given by $P(X = k) = \frac{\lambda^k}{k!}e^{-\cancel{k}\lambda}$ for k=0,1,2,3,...

(a) Calculate the characteristic function $\Phi_X(\omega)$ of X. You may find useful the following: $e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}$.

(b) Calculate the mean and variance of X.

(c) Verity the inequality

$$P(X \le \frac{\lambda}{2}) \le \frac{4}{\lambda}$$

(d) Suppose that Y and Z are also Poisson r.v.s with parameter λ, and that X, Y, Z are independent. What is the distribution of $W = X + Y + Z$?

Solution (a) $\Phi_X(\omega) = \sum_k p(k) e^{j\omega k} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} e^{j\omega k} = \sum_{k=0}^{\infty} \frac{(e^{j\omega}\lambda)^k}{k!} e^{-\lambda}$

$= e^{\lambda e^{j\omega}} e^{-\lambda} = e^{\lambda(e^{j\omega}-1)}$

(b) $\Phi_X'(\omega) = \lambda e^{\lambda(e^{j\omega}-1)} e^{j\omega} j$

$\Phi_X''(\omega) = \lambda e^{\lambda(e^{j\omega}-1)} j e^{j\omega} j + \lambda^2 e^{\lambda(e^{j\omega}-1)} e^{j\omega} e^{j\omega} j j$

$E(X) = \frac{1}{j} \Phi_X'(\omega)|_{\omega=0} = \lambda$ $\quad E(X^2) = \frac{1}{j^2} \Phi_X''(\omega)|_{\omega=0} = \lambda^2 + \lambda$

$Var(X) = E(X^2) - [E(X)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

(c) Apply chebyshev inequality. let $a = \frac{\lambda}{2}$ then

$P[|X-\lambda| \ge \frac{\lambda}{2}] = P[X \ge \frac{3\lambda}{2}] + P[X \le \frac{\lambda}{2}] \le \frac{\lambda}{(\frac{\lambda}{2})^2} = \frac{4}{\lambda}$

therefore $P(X \le \frac{\lambda}{2}) \le \frac{4}{\lambda}$

(d) $\Phi_X(\omega) = \Phi_Y(\omega) = \Phi_Z(\omega) = e^{\lambda(e^{j\omega}-1)}$ then

$\Phi_W(\omega) = \Phi_X(\omega)\Phi_Y(\omega)\Phi_Z(\omega) = e^{3\lambda(e^{j\omega}-1)}$

therefore W are poisson r.v. with parameter 3λ

4. (10 pts) True or False.

Circling the correct answer is worth +2 points, circling the incorrect answer is worth −1 points. Not circling either is worth 0 points.

(a) If X and Y are uncorrelated and exponential then X and Y are independent.

TRUE FALSE

(b) If X is $X \sim N(0,1)$ and $Y = X^2$ then Y is also Gaussian.

TRUE FALSE

(c) Suppose X and Y are independent. Then $E[X^2Y^3] = E[X^2]E[Y^3]$.

TRUE FALSE

(d) Suppose $X \sim N(4,4)$ and $Y \sim N(2,2)$ are X and Y are independent. Let $Z = X - Y$. Then $Z \sim N(2,2)$.

TRUE FALSE

(e) If X and Y are independent and Y and Z are independent, then X and Z are independent.

TRUE FALSE

5. (10+10+5 pts) Consider X and Y be jointly distributed according to the joint density $f(x,y) = k(y-x)^2$, for $0 \le x \le y \le 1$ and zero elsewhere, and where k is a constant.

(a) Find the constant k to make $f_{X,Y}(x,y)$ a valid joint pdf.

(b) Find $f_X(x)$.

(c) Find $f_{X|Y}(x|y)$.

Solution (a) $\iint f(x,y)\,dx\,dy = \int_0^1 \int_0^y k(y-x)^2\,dx\,dy = 1$ solve for k

$$k = \frac{1}{\int_0^1 \int_0^y (y-x)^2\,dx\,dy} = 12$$

(b) $f_X(x) = \int_x^1 f(x,y)\,dy = \int_x^1 12(y-x)^2\,dy = 4(1-x)^3 \quad 0 \le x \le 1$

(c) $f_Y(y) = \int_0^y f(x,y)\,dx = \int_0^y 12(y-x)^2\,dx = 4y^3$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{12(y-x)^2}{4y^3} = \frac{3(y-x)^2}{y^3} \quad 0 \le x \le y \le 1$$

6. (5+5+5+5+5 pts) Suppose X and Y are jointly Gaussian with $E[X] = 4$ and $E[Y] = 2$ and with covariance matrix

$$K_{XY} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

You may find useful the following: if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

(a) What is $f_Y(y)$?

(b) Suppose $Z = 2X + Y$. What is $f_Z(z)$?

(c) Suppose also $W = X - Y$. What is $f_{Z,W}(z, w)$?

(d) What is $f_{X,Z}(X, Z)$?

(e) What is $\rho_{X,Z}$?

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6 Solution

a) $E(Y)=2 \quad Var(Y)=4$ therefore

$$f_Y(y)=\frac{1}{2\sqrt{2\pi}}e^{-\frac{(y-2)^2}{8}}$$

b) $Z=2X+Y$ is also Gaussian r.v.

$$E(Z)=2E(X)+E(Y)=2\cdot 4+2\cdot 1=10$$

$$Var(Z)=4Var(X)+Var(Y)+4Cov(X,Y)=16$$

$$f_Z(z)=\frac{1}{\sqrt{2\pi}\,4}e^{-\frac{(z-10)^2}{32}}$$

c) $\begin{bmatrix} Z \\ W \end{bmatrix}=\begin{bmatrix} 2X+Y \\ X-Y \end{bmatrix}=\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} \qquad A=\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$

$$m_{ZW}=Am_{XY}=\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix}=\begin{bmatrix} 10 \\ 2 \end{bmatrix}$$

$$K_{ZW}=AK_{XY}A'=\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}=\begin{bmatrix} 16 & -1 \\ -1 & 4 \end{bmatrix}$$

$$K_{ZW}^{-1}=\frac{1}{63}\begin{bmatrix} 4 & 1 \\ 1 & 16 \end{bmatrix}$$

$$f_{ZW}=\frac{1}{2\pi\sqrt{63}}e^{-\frac{1}{126}\left[4(z-10)^2+2(z-10)(w-2)+16(w-2)^2\right]}$$

d) $\begin{bmatrix} X \\ Z \end{bmatrix}=\begin{bmatrix} X \\ 2X+Y \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} \qquad A=\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$$m_{XZ}=Am_{XY}=\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix}=\begin{bmatrix} 4 \\ 10 \end{bmatrix}$$

$$K_{XZ}=AK_{XY}A'=\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 2 & 5 \\ 5 & 16 \end{bmatrix}$$

$$K_{XZ}^{-1}=\frac{1}{7}\begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix}$$

$$f_{XZ}(x,z)=\frac{1}{2\pi\sqrt{7}}e^{-\frac{1}{14}\left[16(x-4)^2-10(x-4)(z-10)+2(z-10)^2\right]}$$

e) $\rho_{XZ}=\frac{5}{\sqrt{\ \ }\sqrt{\ \ }}=\frac{5}{4\sqrt{2}}=\frac{5}{8}\sqrt{2}$

7. (15 pts) Let X_1, X_2, X_3... be independent and uniformly distributed on [0,1]. Calculate $\lim_{n\to\infty}\left(\frac{\sin(X_1)+\sin(X_2)+\cdots+\sin(X_n)}{n}\right)$.

Solution X_i be independent and uniformly distributed on [0,1]

Then $E[\sin(X_i)] = \int_0^1 \sin x\,dx = 1-\cos 1$

Apply strong law of large numbers.

$$\lim_{n\to\infty}\frac{\sin X_1+\sin X_2+\cdots+\sin X_n}{n} = E(\sin X_i) = 1-\cos 1$$

8. (10+10 pts) Suppose you have two decks of n cards, numbered 1 through n. The two decks are shuffled and the cards are matched against each other. We say that a match occurs at position i if the ith card from each deck has the same number. Let S_n denote the number of matches.

   (a) Compute $E(S_n)$.

   (b) Compute $VAR(S_n)$.

Solution. (a) Denote X_i equal to 1 if ith card are same equal to 0 if not then $S_n = \sum_{i=1}^{n} X_i$

$$P(X_i) = \begin{cases} \frac{1}{n} & X_i = 1 \\ 1 - \frac{1}{n} & X_i = 0 \end{cases}$$

therefore $E(S_n) = \sum_{i=1}^{n} E(X_i) = n \cdot \frac{1}{n} = 1$

(b)

$$P(X_i X_j) = \begin{cases} \frac{1}{n(n-1)} & X_i X_j = 1 \\ 1 - \frac{1}{n(n-1)} & X_i X_j = 0 \end{cases}$$

$$E(X_i X_j) = \frac{1}{n(n-1)} \qquad E(X_i^2) = \frac{1}{n}$$

$$E[S_n^2] = E[(\sum_{i=1}^{n} X_i)^2] = E[n X_i^2 + (n-1) n X_i X_j]_{i \neq j}$$

$$= n \cdot \frac{1}{n} + (n-1) n \frac{1}{n(n-1)}$$

$$= 2$$

$$Var(S_n) = E(S_n^2) - E^2(S_n) = 2 - 1 = 1$$

9. (20 pts) Let X_1, X_2, X_3... be independent and uniformly distributed on [0,1], and let a be a constant.

Express the following limit

$$\lim_{n\to\infty} P\left(\sqrt{n}\left|\frac{X_1+\ldots+X_n}{n}-\frac{1}{2}\right|>a\right) \tag{1}$$

in terms of $Q(x)$ function (Recall that $Q(x) = P(X > x)$ for X standard Gaussian).

Solution define $Z_n = \frac{X_1+\cdots+X_n}{n}$ then

$E(Z_n) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n} n E(X_i) = \frac{1}{2}$

$Var(Z_n) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{n}{n^2} Var(X_i) = \frac{1}{12n}$

Then $\lim_{n\to\infty} P(\sqrt{n}|Z_n - \frac{1}{2}| > a) = \lim_{n\to\infty} P(|Z_n - \frac{1}{2}| > \frac{a}{\sqrt{n}})$

$= 1 - \lim_{n\to\infty} P(|Z_n - \frac{1}{2}| < \frac{a}{\sqrt{n}})$

$= 1 - \lim_{n\to\infty} P\left(\left|\frac{Z_n - \frac{1}{2}}{\sqrt{\frac{1}{12n}}}\right| < \frac{a}{\sqrt{n\frac{1}{12n}}}\right)$

$= 1 - \lim_{n\to\infty} P\left(\left|\frac{Z_n - \frac{1}{2}}{\sqrt{\frac{1}{12n}}}\right| < 2\sqrt{3}a\right)$

$= 1 - [1 - Q(2\sqrt{3}a)] + [1 - Q(-2\sqrt{3}a)]$

$= 1 + Q(2\sqrt{3}a) - Q(-2\sqrt{3}a)$

$= 2Q(2\sqrt{3}a)$

10. (5+5+10 pts) A communications channel is used to transmit bits of information. Of the transmitted bits, 60% are '1' and 40% are '0'. When the transmitter sends a '1', the input voltage is $v = 5$, and when a '0' is sent the input voltage is $v = -5$. The noise N depends on the input voltage: it is uniformly distributed on $[-8, 8]$ N_1 when $v = 5$ (i.e., when a '1' is sent), and it is uniformly distributed on $[-5, 5]$ when $v = -5$ (i.e., when a '0' is sent). N_2

The decision threshold of the receiver is at a certain voltage T. This means that the receiver at the other end decides that a '1' was sent if $v + N > T$, and that a '0' was sent if $v + N \leq T$.

(a) Find the probability that the receiver decides a '1' was sent, given that the transmitter sent a '0' for the case where $T = 0$.

(b) Find the total probability of the receiver error for $T = 0$.

(c) Find the value of the decision threshold T which minimizes the total probability of receiver error.

Solution (a) Define R as receive and S as send

$$P[R=1|S=0] = P[V+N_2 \geq 0 | V=-5] = P[N_2 \geq 5] = 0$$

Since N_2 is uniformly distributed in $[-5, 5]$

(b) $P[\text{error}] = P[R=1|S=0]P[S=0] + P[R=0|S=1]P[S=1]$

$= 0 + P[V+N_1 \leq 0 | V=5] \cdot 0.6$

$= P[5+N_1 < 0] \cdot \frac{3}{5}$ N_1 is uniformly distributed in $[-8, 8]$

$= \frac{-5-(-8)}{8-(-8)} \cdot \frac{3}{5}$ [illegible]

(c) $P[R=1|S=0] = P[-5+N_2 > T] = P[N_2 > T+5] = \begin{cases} 1 & \text{if } T < -10 \\ -T/10 & \text{if } -10 \leq T < 0 \\ 0 & \text{if } T \geq 0 \end{cases}$

$P[R=0|S=1] = P[N_1+5 \leq T] = P[N_1 \leq T-5] = \begin{cases} 0 & \text{if } T < -3 \\ \frac{T+3}{16} & \text{if } -3 \leq T < 13 \\ 1 & \text{if } T \geq 13 \end{cases}$

$P[\text{error}] = P[R=1|S=0]P[S=0] + P[R=0|S=1]P[S=1]$

The overall minimum of error is obtained when $T=0$

$$= \begin{cases} 0.4 & \text{if } T < -10 \\ -T/25 & \text{if } -10 \leq T < -3 \\ -T/25 + 0.6(T+3)/16 & \text{if } -3 \leq T < 0 \\ 0.6(T+3)/16 & \text{if } 0 \leq T < 13 \\ 0.6 & \text{if } T \geq 13 \end{cases}$$