Midterm Exam Solutions

1. (20 points) Signal processing foundations

a) (5 points) Assume that $x(n)$ and $h(n)$ are discrete signals specified as follows:

$x(n) = [\ \underline{2}\ \ -3\ \ 1\ \ 2]$ $h(n) = [\ \underline{-4}\ \ 2\ \ 1]$

All values not explicitly specified are assumed to be zero. As is customary, the underbar indicates the location of the origin. Find $y(n)$, where $y(n) = x(n) * h(n)$. Be sure to clearly indicate the location of the origin in your answer.

Solution: $y(n) = [\ \underline{-8}\ \ 16\ \ -8\ \ -9\ \ 5\ \ 2]$

b) (15 points) Consider a 1D discrete system that is known to be linear. Suppose, further, that you know that:

An input $x_1(n)=[3\ \ \underline{1}]$ produces output $y_1(n)=[\underline{3}\ \ 4\ \ 1]$

An input $x_2(n)=[3\ \ \underline{1}\ \ 1]$ produces output $y_2(n)=[\underline{3}\ \ 6\ \ 2]$

An input $x_3(n)=[\ \underline{-4}\ \ 1]$ produces output $y_3(n)=[\underline{0}\ \ -2\ \ -3]$

All values not explicitly specified are assumed to be zero. The underbar indicates the location of the origin. State, with a proof, whether or not this is a time invariant system.

Solution

Since the system is linear, we can find the output for any linear combination of $x_1(n)$, $x_2(n)$, and $x_3(n)$.

In particular, let $x_4(n)= x_2(n)-x_1(n)=\delta(n-1)$. Then, $y_4(n)= y_2(n)-y_1(n)$)=$[\underline{0}\ \ 2\ \ 1]$.

Thus, an input $\delta(n-1)$ gives an output $[\underline{0}\ \ 2\ \ 1]$.

Next, we can construct an input $\delta(n)$ by combining $x_3(n)$ and $-\delta(n-1)$:

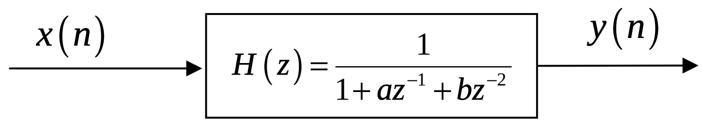
$x_5(n)= \delta(n) = -(1/4)[x_3(n)-\delta(n-1)]$ gives an output
$y_5(n) = -(1/4)[y_3(n) - [\underline{0}\ \ 2\ \ 1]\ ]= [\underline{0}\ \ 1\ \ 1]$

Thus, an input $\delta(n)$ gives an output $[\underline{0}\ \ 1\ \ 1]$.

Since inputs $\delta(n)$ and $\delta(n-1)$ give outputs that do not differ only by a shift of one unit, the system is not be time invariant. Or, in other words, the system is time-variant.

2. (30 points) Z-Transform

Consider a discrete time two-pole filter depicted below:



The values a and b are real constants.

a) (5 points) Assume that $a = -1, b = \frac{1}{2}$. Find the location of the poles in z domain.

Solution

$$H(z) = \frac{1}{1 - z^{-1} + \frac{1}{2}z^{-2}} = \frac{1}{\left(1 - \left(\frac{1}{2} + \frac{1}{2}j\right)z^{-1}\right)\left(1 - \left(\frac{1}{2} - \frac{1}{2}j\right)z^{-1}\right)}$$

$$p_1 = \frac{1}{2} + \frac{1}{2}j, \; p_2 = \frac{1}{2} - \frac{1}{2}j$$

b) (10 points) Suppose that the sampling frequency is 1000 Hertz and that $a = -1, b = \frac{1}{2}$.
Find $|H(w)|$, the magnitude response of the filter, at a frequency of 250 Hz.

Solution

First compute the normalized frequency ω corresponding to 250 Hz:
$w = \frac{250}{1000} \cdot 2p = \frac{p}{2}$ (note that there will also be an component at ω = - π/2, but the magnitude response will be the same as at π/2)

Next, to obtain $|H(w)|$ from $H(z)$, evaluate $H(z)$ at $z = e^{jw}$ and then take the absolute value:

$$H\left(z = e^{j\frac{p}{2}}\right) = H(z = j) = \frac{1}{1 - j^{-1} + \frac{1}{2} j^{-2}} = \frac{1}{1 + j - \frac{1}{2}} = \frac{1}{\frac{1}{2} + j}$$

$$\left|H\left(w = \frac{p}{2}\right)\right| = \left|H(z = j)\right| = \frac{1}{\sqrt{1 + \frac{1}{4}}} = \frac{1}{\sqrt{\frac{5}{4}}} = \frac{2}{\sqrt{5}}$$

c) (15 points) Assume that the parameter b is set to a fixed, known constant value (but not necessarily ½). Suppose that you have the freedom to select the value of a.

Find the value of a that will result in a filter with an amplitude response with a peak at F_1 Hz, where $0 < F_1 < \frac{F_S}{2}$? Assume a sampling frequency denoted by F_S.

Solution

Using the quadratic formula, the poles will be at the locations:

$$\frac{-a \pm \sqrt{a^2 - 4b}}{2} = \frac{-a \pm \sqrt{-1}\sqrt{4b - a^2}}{2} = \frac{-a \pm j\sqrt{4b - a^2}}{2}$$

When there are complex conjugate poles, $a^2 - 4b$ will be smaller than zero, and the poles will have a real coordinate $\frac{-a}{2}$ and an imaginary coordinate $\frac{\pm j\sqrt{4b - a^2}}{2}$. Thus, the two complex conjugate poles will be at a radius of

$\sqrt{\frac{a^2}{4} + \frac{4b - a^2}{4}} = \sqrt{b}$. To create a peak at F_1 Hertz, poles should be placed at angular positions given by $w = \pm 2p \frac{F_1}{F_S}$.

Combining the above means that the poles are at $\sqrt{b}e^{\pm j2p\frac{F_1}{F_S}}$. The real coordinate of the pole location is $\sqrt{b}\cos 2p \frac{F_1}{F_S}$. From above, we know this is equal to $\frac{-a}{2}$, so this means that $a = -2\sqrt{b}\cos 2p \frac{F_1}{F_S}$.

3. (25 points) Speech filter design

Consider a speech signal that has been sampled at 8 KHz.

a) (7 points) In this system, what is the normalized frequency ω corresponding to 2 KHz?

Solution
Using the correspondence that the sampling frequency F_s maps to $\omega=2\pi$, a frequency of 2 KHz corresponds to $\omega = \pm\,\pi/2$.

b) (18 points) Design a causal filter that will completely attenuate a steady state a sinusoidal signal at 2 KHz. Your answer should be expressed as an impulse response h(n), expressed in simplified form, with the location of the origin clearly noted, and all nonzero values of h(n) are clearly specified in maximally simplified form. As a reminder, "causal" means h(n) is zero for n<0.

Solution
Placing zeros on the z plane at z = j and z = -j, which corresponds to $\omega = \pm\,\pi/2$ on the unit circle, will completely attenuate signals at 2 KHz.

$H(z)=(z-j)(z+j)=z^2+1$. The inverse z transform of this is h(n) = [1 0 $\underline{1}$]. However, this is not causal. To make it causal, we can delay h(n) by two units, because shifts in h(n) will not impact the magnitude of $H(w)$ (we know this from the Fourier shift theorem). Thus, a filter meeting the desired goals is:

$H(z)=(z^{-1}-j)(z^{-1}+j)=1+z^{-2}$, which has inverse z transform h(n) = [$\underline{1}$ 0 1] (where all values not explicitly shown are zero).

To confirm that we have obtained the desired frequency response, we can obtain H(ω):

$H(w)=H(z)$ evaluated at $z=e^{jw}$
$H(w)=1+e^{-2jw}$

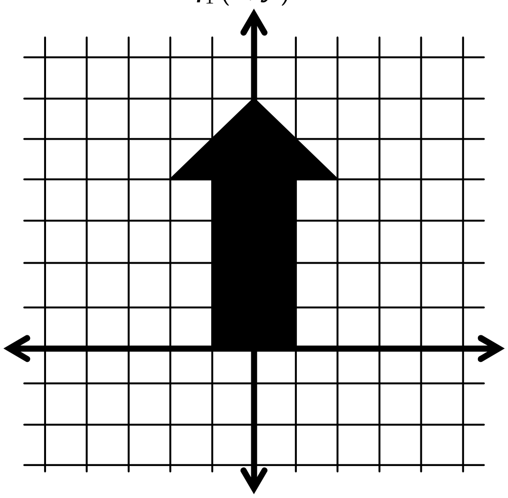
$$|H(w)|^2=(1+e^{-2jw})(1+e^{2jw})=2+2\cos(2w)$$

Plugging $\omega = \pm\,\pi/2$ into this equation gives 0. Note that the result can also be confirmed by evaluating $H(w)$ (instead of $|H(w)|^2$) at $\omega = \pm\,\pi/2$, which also gives 0. Note that impulse responses of the form h(n) = A[$\underline{1}$ 0 1], where A is any constant, are also valid.

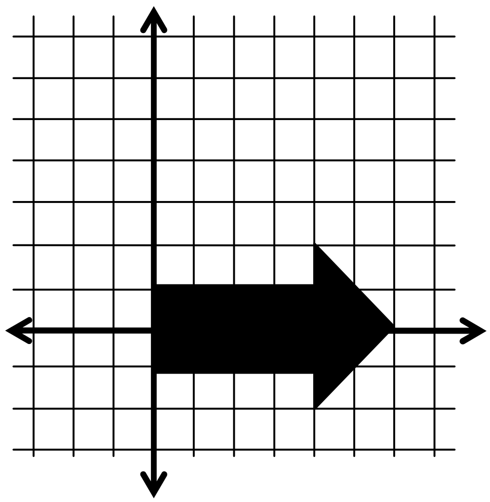
4. 2D Convolution (25 points)

Consider the two continuous 2D functions $f_1(x,y)$ and $f_2(x,y)$ given below, which have value 1 where the dark regions are located and 0 elsewhere. Make a sketch showing the boundaries of non-zero regions of 2D function obtained by the convolution of these two functions, i.e. $f_1 * f_2$. Be sure to place your sketch on a set of marked axis such that relevant dimensions and positions of features are clearly indicated.

$f_1(x,y)$



$f_2(x,y)$



Solution