Solutions

1. (10 points) Consider the following 4×4 image, where the numbers represent levels of gray in the range 0 (black) through 7 (white):

1	2	2	1
2	4	4	2
2	4	4	2
1	2	2	1

Apply histogram equalization to the original image, where the target cumulative histogram is linear, starting at 2 at gray level 0. Find the resulting image and the corresponding histogram. Has contrast been enhanced and how can you tell?

Solution: Histogram and cumulative histogram of the original image:

u	$p(u)$	$P(u)$
0	0	0
1	4	4
2	8	12
3	0	12
4	4	16
5	0	16
6	0	16
7	0	16

Desired cumulative histogram:

y	$P_{\mathrm{d}}(y)$
0	2
1	4
2	6
3	8
4	10
5	12
6	14
7	16

Transformation:

u	y
0	0
1	1
2	5
3	5
4	7
5	7
6	7
7	7

Output image:

1	5	5	1
5	7	7	5
5	7	7	5
1	5	5	1

Histogram and cumulative histogram of the output image:

y	$p(y)$	$P(y)$
0	0	0
1	4	4
2	0	4
3	0	4
4	0	4
5	8	12
6	0	12
7	4	16

Before processing, the darkest pixels were at gray level 1 and the brightest ones at gray level 4; after processing, the darkest pixels are still at gray level 1, but the brightest pixels are now at gray level 7. Contrast has been enhanced.

2. (a) (5 points) Construct a 3-point unitary DCT matrix. Your matrix should contain only exact numbers (i.e. **NO** functions and decimals). Your matrix should look like the one below.

$$\boldsymbol{C} = \frac{1}{\sqrt{3}}\begin{bmatrix} \square & \square & \square \\ \square & 0 & \square \\ \square & \square & \square \end{bmatrix}$$

Solution:

$$C(k,n) = \alpha(k)\cos\left(\frac{\pi(2n+1)k}{6}\right)$$

Where

$$\alpha(k) = \begin{cases} \frac{1}{\sqrt{3}} & \text{if } k=0 \\ \sqrt{\frac{2}{3}} & \text{if } k=1,2 \end{cases}$$

Therefore,

$$\boldsymbol{C} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}}\frac{\sqrt{3}}{2} & 0 & -\sqrt{\frac{2}{3}}\frac{\sqrt{3}}{2} \\ \sqrt{\frac{2}{3}}\frac{1}{2} & -\sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}}\frac{1}{2} \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ \sqrt{\frac{3}{2}} & 0 & -\sqrt{\frac{3}{2}} \\ \frac{1}{\sqrt{2}} & -\sqrt{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

(b) (10 points) Find a transformation matrix $\boldsymbol{\Phi}$ transforming $\boldsymbol{V}_{\mathrm{C}}$ to $\boldsymbol{V}_{\mathrm{F}}$, where $\boldsymbol{V}_{\mathrm{C}}$ is the unitary 3-DCT of $\boldsymbol{u}$ and $\boldsymbol{V}_{\mathrm{F}}$ is the unitary 3-DFT of $\boldsymbol{u}$. In other words, let $\boldsymbol{u} \xrightarrow{\text{DCT}} \boldsymbol{V}_{\mathrm{C}}$ and $\boldsymbol{u} \xrightarrow{\text{DFT}} \boldsymbol{V}_{\mathrm{F}}$. Find a 3-by-3 matrix describing the operation $\boldsymbol{V}_{\mathrm{C}} \to \boldsymbol{V}_{\mathrm{F}}$. Your matrix should look like the one below.

$$\boldsymbol{\Phi} = \frac{1}{3}\begin{bmatrix} \square & \square & \square \\ \square & \sqrt{\frac{3}{2}}\left(\frac{3}{2}\square \mathrm{j}\frac{\sqrt{3}}{2}\right) & \frac{1}{\sqrt{2}}\left(\frac{3}{2}\square \mathrm{j}\frac{3\sqrt{3}}{2}\right) \\ \square & \sqrt{\frac{3}{2}}\left(\frac{3}{2}\square \mathrm{j}\frac{\sqrt{3}}{2}\right) & \frac{1}{\sqrt{2}}\left(\frac{3}{2}\square \mathrm{j}\frac{3\sqrt{3}}{2}\right) \end{bmatrix}$$

Solution: Since $\boldsymbol{V}_{\mathrm{C}} = \boldsymbol{C}\boldsymbol{u}$ and $\boldsymbol{V}_{\mathrm{F}} = \boldsymbol{F}\boldsymbol{u}$, $\boldsymbol{u} = \boldsymbol{C}^{\mathsf{T}}\boldsymbol{V}_{\mathrm{C}}$ and $\boldsymbol{V}_{\mathrm{F}} = (\boldsymbol{F}\boldsymbol{C}^{\mathsf{T}})\boldsymbol{V}_{\mathrm{C}}$. Therefore,

$$\begin{aligned} \boldsymbol{\Phi} = \boldsymbol{F}\boldsymbol{C}^{\mathsf{T}} &= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} - \mathrm{j}\frac{\sqrt{3}}{2} & -\frac{1}{2} + \mathrm{j}\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} + \mathrm{j}\frac{\sqrt{3}}{2} & -\frac{1}{2} - \mathrm{j}\frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 1 & \sqrt{\frac{3}{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & -\sqrt{2} \\ 1 & -\sqrt{\frac{3}{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}}\left(\frac{3}{2} - \mathrm{j}\frac{\sqrt{3}}{2}\right) & \frac{1}{\sqrt{2}}\left(\frac{3}{2} + \mathrm{j}\frac{3\sqrt{3}}{2}\right) \\ 0 & \sqrt{\frac{3}{2}}\left(\frac{3}{2} + \mathrm{j}\frac{\sqrt{3}}{2}\right) & \frac{1}{\sqrt{2}}\left(\frac{3}{2} - \mathrm{j}\frac{3\sqrt{3}}{2}\right) \end{bmatrix} \end{aligned}$$

(c) (10 points) Let $V_{\mathrm{F}}(k) = \mathrm{DFT}\{u(n)\} = \begin{bmatrix}\underline{0} & 1 & 1\end{bmatrix}$. Find $V_{\mathrm{C}}(k) = \mathrm{DCT}\{u(n)\}$.

Solution: The matrix product of two unitary matrices is still unitary. One can easily verify by checking the norm of each column and the inner product between each column in $\boldsymbol{\Phi}$. Using the property of unitary matrix we get

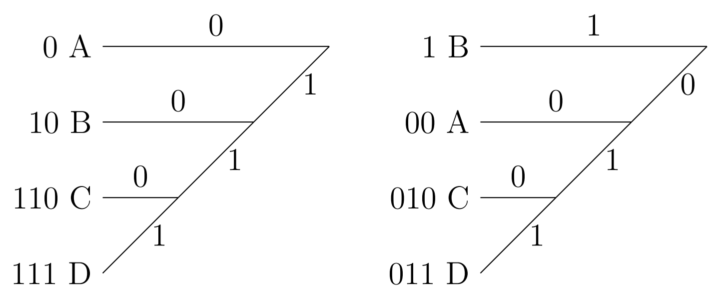
$$\boldsymbol{V}_{\mathrm{C}} = \boldsymbol{\Phi}^{\mathsf{H}}\boldsymbol{V}_{\mathrm{F}} = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}}\left(\frac{3}{2} + \mathrm{j}\frac{\sqrt{3}}{2}\right) & \sqrt{\frac{3}{2}}\left(\frac{3}{2} - \mathrm{j}\frac{\sqrt{3}}{2}\right) \\ 0 & \frac{1}{\sqrt{2}}\left(\frac{3}{2} - \mathrm{j}\frac{3\sqrt{3}}{2}\right) & \frac{1}{\sqrt{2}}\left(\frac{3}{2} + \mathrm{j}\frac{3\sqrt{3}}{2}\right) \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{\frac{3}{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

3. (a) (10 points) Consider a set of symbols, A, B, C, and D, with associated probabilities $P(A), P(B), P(C)$ and $P(D)$. Suppose that two different Huffman codes for this symbol set are designed, using standard Huffman code design procedures.

	Code 1	Code 2
A	0	00
B	10	1
C	110	010
D	111	011

If $P(B) = \frac{2}{5}$, find $P(C) + P(D)$.

Solution: We first construct two trees from the codes above.



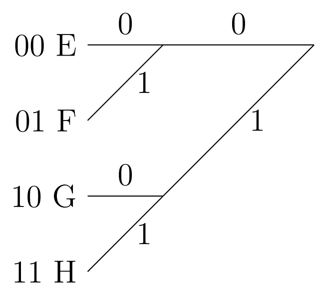
Note that node $(C + D)$ can connect to either node first A or node B, which means $P(A) = P(B) = \frac{2}{5}$. Therefore, $P(C) + P(D) = 1 - P(A) - P(B) = 1 - \frac{2}{5} - \frac{2}{5} = \boxed{\frac{1}{5}}$.

(b) (10 points) Consider another set of symbols, E, F, G, and H, with associated probabilities $P(E), P(F), P(G)$ and $P(H)$. Suppose that three different Huffman codes for this symbol set are designed, using standard Huffman code design procedures.

	Code 1	Code 2	Code 3
E	0	00	00
F	10	01	1
G	110	10	010
H	111	11	011

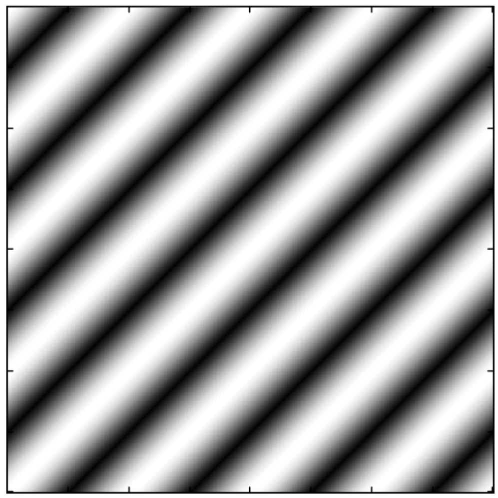
If $P(G) = \frac{1}{9}$, find $P(H)$.

Solution: Code 1 and Code 3 generate the same Huffman trees as the previous ones in part (a), which means $P(E) = P(F)$. The tree generated from Code 2 is shown below.



The additional tree shows that $P(E) = P(F) = P(G) + P(H) = \frac{1}{3}$ since the three nodes $P(E)$, $P(F)$, and $P(G) + P(E)$ can be connected in all possible orders. Since $P(G) = \frac{1}{9}$, $P(H) = \frac{1}{3} - P(G) = \frac{1}{3} - \frac{1}{9} = \boxed{\dfrac{2}{9}}$.

4. A 2D continuous sinusoidal wave of unit amplitude $f_{\mathrm{a}}(x, y)$ propagating along the diagonal direction is sampled (above Nyquist rates) at $F_{\mathrm{s}x} = F_{\mathrm{s}y} = 8000$ pixels/cm, to obtain $f(m, n) = f_{\mathrm{a}}(m/F_{\mathrm{s}x}, n/F_{\mathrm{s}y})$. The magnitude of the resulting image, $|f(m, n)|$, is shown below in an 800×800 pixel frame, where black represents 0 and white 1.



Remember that the origin is located at the upper-left corner. Also, m corresponds to the horizontal coordinate and n corresponds to the vertical coordinate.

(a) (5 points) What are the spatial frequencies of the analog sinusoid in the horizontal and vertical direction, F_x and F_y, respectively?

Solution: The analog sinusoid is

$$f_{\mathrm{a}}(x, y) = \cos\left(2\pi F_x x + 2\pi F_y y\right),$$

where $1/F_x$ and $1/F_y$ are its horizontal and vertical periods. The sampled image is equal to

$$f(m, n) = \cos\left(2\pi \frac{F_x}{F_{\mathrm{s}x}} m + 2\pi \frac{F_y}{F_{\mathrm{s}y}} n\right).$$

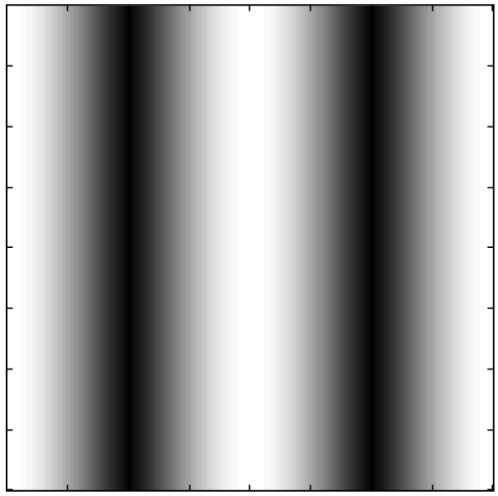
From the figure we see that the periods of $|f(m, n)|$ along each direction are equal to $\frac{800}{4} = 200$ pixels, which means that the periods of $f(m, n)$ are 400 pixels in each direction. This implies that

$$F_x = \frac{F_{\mathrm{s}x}}{400} = 20\,\mathrm{cm}^{-1}, \quad F_y = F_x.$$

This means that

$$f_{\mathrm{a}}(x, y) = \cos\left(2\pi(20x + 20y)\right).$$

(b) (10 points) Design a new set of (non-zero!) sampling rates (F_{sx_new}, F_{sy_new}) for which the resampled image, $f_r(m, n)$, is such that $|f_r(m, n)|$ will look like the image below in an 800×800 pixel frame. Will the image reconstructed from $f_r(m, n)$ be aliased?



Solution: The resampled image is equal to

$$f_r(m, n) = \cos\left(2\pi \frac{F_x}{F_{sx\_new}} m + 2\pi \frac{F_y}{F_{sy\_new}} n\right).$$

From the figure, it appears that the resampled image is a sinusoid with a period in the horizontal direction of 800 pixels and a (seemingly) zero frequency in the vertical direction. This yields

$$\frac{F_{sx\_new}}{20} = 800 \quad \Rightarrow \quad F_{sx\_new} = 16\,000\,\text{pixels/cm}, \quad F_{sy\_new} = \frac{F_y}{p},$$

for any positive integer p. Let us take $p = 1$, which means $F_{sy\_new} = 20\,\text{pixels/cm}$. Clearly, F_{sy_new} is below the Nyquist rate in the vertical direction and the reconstructed image will be aliased.