EE 113

Final Review Summary

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This is just a brief summary of what I went over in my discussions and office hours. I went over many problems from old finals, which I cannot write up the solutions to for lack of time. But I am giving the problems with answers for most and I will try to post some hints later today or tomorrow (Sunday 03/19/06).

Summary of some relationships among various Transforms

Given a signal $x(n)$,

Z-Transform: $X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$, ROC needs to be specified.

DTFT: $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-in\omega}$

$X(\omega) = X(z)|_{z=e^{i\omega}}$ if ROC of $X(z)$ includes the unit circle $\{|z| = 1\}$.

Exceptions: $x(n) = 1$ doesn't have a ZT. But $X(\omega)$ exists

$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k2\pi)$

DFT: If $x(n)$ is a finite signal, between $0 \le n \le N-1$, the DFT is

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-i\frac{2\pi nk}{N}},\ 0 \le k \le N-1$$

Clearly

$$X(k) = X(\omega)|_{\omega=\frac{2\pi k}{N}} = X(z)\big|_{z=e^{i\frac{2\pi k}{N}}}$$

($X(z)$ and $X(\omega)$ will always exist for a finite signal).

FFT: Just a fast algorithm for computing the DFT for highly composite N.

Some symmetry properties: If $x(n)$ is real, then $X(z^*) = X^*(z)$ (See lecture notes Section 5, page 4 for proof).

Corresponding symmetry properties for DTFT and DFT:

DTFT: $X(-\omega) = X^*(\omega)$ if $x(n)$ real.

DTFT: $x(n)$ real $\Rightarrow X(N-k) = X^*(k), 1 \le k \le N-1$ (for all k if $X(k)$ is extended periodically).

(For the DTFT the property follows by letting $z = e^{i\omega}$ for signals that have a Z-transform and follows for the DFT with $\omega = 2\pi k/N$ in the DTFT).

Problem from old final: If x(n) is real and even (i.e., $x(n) = x^*(n)$ and $x(-n) = x(n)$),

(i) Show that if $X(z_0) = 0$ for some complex number z_0, then we also have $X(1/z_0) = 0$.

(ii) Are there any other zeros implied by the information given? (Hint: think complex conjugates).

Note that the proof of (i) above, you will end up showing that if $x(n)$ is even, actually $X(1/z) = X(z)$.

Can you see how this would imply that $X(z^*) = X(z)$ if $|z| = 1$? (Basically because $z^* = 1/z$ for $|z| = 1$).

Which is basically a symmetry property for the DTFT: If $x(n)$ is real and even, the DTFT $X(\omega)$ is real and even. (i.e. $X(-\omega) = X(\omega)$ and $X(\omega) = X^*(\omega)$).

What would be a corresponding symmetry property for the DFT? (Basically if $x(N-n) = x(n)$, $1 \le n \le N-1$, then $X(k)$ is real and so $X(N-k) = X(k), 1 \le k \le N-1$.)

Some problems from old finals:

Problem 1. Suppose $H(z)$, the Z-Transform of $h(n)$ is

$$H(z) = \frac{z}{z - \frac{1}{2}}, \ |z| > \frac{1}{2}$$

and you are given the sequence

$$h_2(n) = \begin{cases} nh(n), & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

Find $H_2(z)$.

Answer:

$$H_2(z) = \frac{\frac{1}{2}z^2}{(z^2 - \frac{1}{4})^2}, \ |z| > \frac{1}{2}$$

Problem 2. Compute the energy in the following N-point sequence.

$$x(n) = \frac{1}{2}\sin(2\pi k_0 n/N),\ 0 \le n \le N-1$$

where k_0 is an integer.

Answer: If N is even,

$$E = \begin{cases} 0, & k_0 = 0 \mod N/2 \\ \frac{N}{8}, & k_0 \neq 0 \mod N/2 \end{cases}$$

If N is odd,

$$E = \begin{cases} 0, & k_0 = 0 \mod N \\ \frac{N}{8}, & k_0 \neq 0 \mod N \end{cases}$$

Hint: Recall formula for sine resulting from Euler's formula: $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$. Recall the *derivation* of the important equation from the DFT material:

$$\sum_{n=0}^{N-1} e^{\pm i\frac{2\pi kn}{N}} = N\delta(k \mod N) = \begin{cases} N, & k = 0 \mod N \\ 0 & \text{otherwise} \end{cases}$$

Problem 3. Compute $X(z)$ where

$$x(n) = \begin{cases} \alpha^n u(n), & n \text{ is a multiple of } k \\ \beta^n u(n), & \text{elsewhere} \end{cases}$$

where k is a positive integer.

Answer:

$$X(z) = \frac{1}{1-\alpha^k z^{-k}} + \frac{1}{1-\beta z^{-1}} - \frac{1}{1-\beta^k z^{-k}},\ |z| > \max\{\alpha, \beta\}$$

Problem 4. Suppose we receive the analog signal

$$r_a(t) = A\cos(2\pi f t + \pi/4)$$

Here the amplitude A is a constant but we do *not* know its value. Also we do *not* know what the analog frequency is and we do *not* know the phase is $\pi/4$.

Suppose someone takes the Fourier transform of the signal and by looking at the spectrum estimates the frequency to be 47 Hz. However, suppose the actual frequency is 50 Hz. Using the 47 Hz estimate for f, describe and implement an algorithm for estimating the amplitude A. Explain analytically the effect of using the wrong frequency in your algorithm. If you use integration in your algorithm does the frequency error affect how long you might integrate? If so, explain why?

Problem 5. A certain LTI system of the form

$$H(z) = \frac{(z - z_0)(z - z_1)}{(z - p_0)(z - p_1)}$$

is known to have the unit circle inside its ROC. It is known that $z_0 = 0$ and $z_1 = 1/2$. The locations of the poles are unknown, but it is known that they form a complex conjugate pair.

A test is performed and it is determined that the frequency response is $3/4$ at both $\omega = 0$ and $\omega = \pi$.

- Find the locations of the two poles in the z-plane
- Find the difference equation that corresponds to this system. If you are unable to work part (a) then let $p_0 = \frac{1}{5} + i\frac{\sqrt{3}}{5}$ (not the right answer) to work this part.

Answer:

$$p_{0,1} = \frac{1}{3} \pm i\frac{\sqrt{2}}{3}$$

The difference equation is:

$$y(n) - \frac{2}{3}y(n-1) + \frac{1}{3}y(n-2) = x(n) - \frac{1}{2}x(n-1),$$

Problem 6. Consider the causal system described by the following difference equation:

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n),$$
$$y(-1) = 0,\ y(-2) = 8,\ x(n) = (1/3)^n\, u(n)$$

Compute $y(n)$.

(Extra problem: Identify the z-transforms of the zero-state and zero-input responses).

Answer:

$$y(n) = \left[4\left(\frac{1}{2}\right)^n + 4\left(\frac{1}{4}\right)^n - 8\left(\frac{1}{3}\right)^n\right] u(n)$$

Problem 7. Signal reconstruction by linear interpolation between samples. Given an analog signal $x_a(t)$, which is sampled with a period T to get $x(n) = x_a(nT)$, let the signal $\hat{x}_a(t)$ be reconstructed as:
For $-\infty \le n \le \infty$,

$$\hat{x}_a(t) = \frac{1}{T}\left[x_a(nT) + \frac{x_a((n+1)T) - x_a(nT)}{T}(t - nT)\right],\ nT \le t \le (n+1)T$$

Show that

$$\hat{X}_a(\Omega) = X(\omega)|_{\omega=\Omega T} \cdot \frac{\sin^2(\Omega T/2)}{(\Omega T/2)^2}$$

where $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-i\omega n} = \sum_{n=-\infty}^{\infty} x_a(nT)e^{-i\omega n}$