EE 113 Midterm Solution

Winter 2007

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Problem	Points	Score
1	9	
2	15	
3	18	
4	18	
5	10	
6	10	
7	10	
8	10	
Total	100	

EE113 Winter 2007 Midterm Solution (2/14/2007)

Problem 1. For parts (a), (b) and (c) determine whether or not the system is

i. linear

ii. time-invariant

iii. BIBO stable, i.e., bounded input-bounded output stable

a. $y(n) = x(n) + 1$.

b. $y(n) = \cos^2[x(n)] + \sin^2[x(n)]$.

c. $y(n) = n[x(n)]$.

Note: You do not need to show a lot of work on this problem if you can quickly recognize the answer.

(a) Non-linear

$$\because \quad T[ax_1(n) + bx_2(n)] = ax_1(n) + bx_2(n) + 1$$
$$\neq aT[x_1(n)] + bT[x_1(n)] = ax_1(n) + bx_2(n) + a + b$$

Time-invariant

$$\because \quad T[x(n-k)] = x(n-k) + 1 = y(n-k)$$

BIBO stable

$$\because \quad \textit{for bounded input} \quad |x(n)| \le M < \infty$$
$$|y(n)| = |x(n) + 1| \le |x(n)| + 1 \le M + 1 = N < \infty \quad \textit{which is also bounded}$$

(b) Non-linear

$$\because \quad T[ax_1(n) + bx_2(n)] = 1 \neq aT[x_1(n)] + bT[x_1(n)] = a + b$$

Time-invariant

$$\because \quad T[x(n-k)] = 1 = y(n-k)$$

BIBO stable

$\because \quad for \quad bounded \quad input \quad |x(n)| \le M < \infty$

$|y(n)| = 1 < \infty \quad which \quad is \quad also \quad bounded$

(c) Linear

$$\because \quad T[a \cdot x_1(n) + b \cdot x_2(n)] = n \cdot (a \cdot x_1(n) + b \cdot x_2(n))$$
$$= a \cdot T[x_1(n)] + b \cdot T[x_1(n)] = n \cdot a \cdot x_1(n) + n \cdot b \cdot x_2(n)$$

Time-variant

$$\because \quad T[x(n-k)] = n \cdot x(n-k) \ne (n-k) \cdot x(n-k) = y(n-k)$$

Not BIBO stable

$\because \quad for \quad bounded \quad input \quad |x(n)| \le M < \infty$

$|y(n)| = |n \cdot x(n)| = |n| \cdot |x(n)| \to \infty \quad when \quad n \to \infty \quad which \quad is \quad not \quad bounded$

Problem 2. Consider the system described by the following difference equation:

$$y(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = x(n),$$

where,

$$x(n) = (1/4)^n\, u(n), \quad y(-1) = 1, \; y(-2) = 0.$$

a. Find a closed form expression for $y(n)$ using any method we discussed in class.

b. Evaluate your $y(n)$ for $n = 0, 1, 2$.

(a) We can use unilateral Z-transform to solve this problem.

Perform the unilateral Z-transform on the above LCCDE (linear constant coefficient difference equation), we got

$$Y^{+}(Z)-\frac{5}{6}Z^{-1}\left[Y^{+}(Z)+y(-1)\cdot Z\right]+\frac{1}{6}Z^{-2}\left[Y^{+}(Z)+y(-1)\cdot Z+y(-1)\cdot Z^{2}\right]=X^{+}(Z) \quad -- \quad (1)$$

$\because y(-1)=1, y(-2)=0$ *and* $X^{+}(Z)=\frac{Z}{Z-4}$, (1) *can be rearranged to*

$$(1-\frac{5}{6}Z^{-1}+\frac{1}{6}Z^{-2})\cdot Y^{+}(Z)=\frac{5}{6}-\frac{1}{6}Z^{-1}+\frac{Z}{Z-4}, \quad \textit{therefore}$$

$$Y^{+}(Z)=\frac{\frac{5}{6}-\frac{1}{6}Z^{-1}+\frac{Z}{Z-4}}{1-\frac{5}{6}Z^{-1}+\frac{1}{6}Z^{-2}} = \frac{11}{6}+\frac{\frac{15}{4}}{Z-\frac{1}{2}}+\frac{-\frac{26}{9}}{Z-\frac{1}{3}}+\frac{\frac{3}{4}}{Z-\frac{1}{4}}$$

$$=\frac{11}{6}+\frac{\frac{15}{4}}{Z-\frac{1}{2}}+\frac{-\frac{26}{9}}{Z-\frac{1}{3}}+\frac{\frac{3}{4}}{Z-\frac{1}{4}} = \frac{11}{6}+\frac{15}{4}Z^{-1}\frac{Z}{Z-\frac{1}{2}}-\frac{26}{9}Z^{-1}\frac{Z}{Z-\frac{1}{3}}+\frac{3}{4}Z^{-1}\frac{Z}{Z-\frac{1}{4}}$$

Therefore, $y(n)=\frac{11}{6}\delta(n)+[\frac{15}{4}\cdot\left(\frac{1}{2}\right)^{n-1}-\frac{26}{9}\cdot\left(\frac{1}{3}\right)^{n-1}+\frac{3}{4}\cdot\left(\frac{1}{4}\right)^{n-1}]\cdot u(n-1)$

(b) $y(0)=\frac{11}{6}$, $\quad y(1)=\frac{29}{18}$, $\quad y(2)=\frac{475}{432}$

Problem 3. Compute $X(z)$, the forward z-transform, (if it exists) for each of the following. Remember to specify the region of convergence in each case. If the forward z-transform does not exist, explain why.

a. $x(n)=\left(\frac{1}{3}\right)^{n}u(n-1)$.

b. $x(n)=nu(n-1)$.

c.

$$x(n)=\begin{cases}\alpha^{n}u(n), & n \text{ is a multiple of } 2,\\ 0, & \text{elsewhere.}\end{cases}$$

(a)

$$X(Z) = \sum_{n=-\infty}^{\infty}\left(\frac{1}{3}\right)^{n}\cdot u(n-1)\cdot Z^{-n} = \sum_{n=1}^{\infty}\left(\frac{1}{3}\cdot Z^{-1}\right)^{n} = \frac{\frac{1}{3}Z^{-1}}{1-\frac{1}{3}\cdot Z^{-1}} = \frac{\frac{1}{3}}{Z-\frac{1}{3}} \quad if \quad \left|\frac{1}{3}\cdot Z^{-1}\right| < 1$$

$$Therefore \quad X(Z) = \frac{\frac{1}{3}}{Z-\frac{1}{3}}, \quad R.O.C \quad |Z| > \frac{1}{3}$$

(b)

$$Let \quad x_1(n) = u(n), \quad x_2(n) = x_1(n-1) = u(n-1)$$

$$then \quad x(n) = n\cdot x_2(n)$$

$$X(Z) = -Z\cdot\frac{d}{dZ}\left(X_2(Z)\right) = -Z\cdot\frac{d}{dZ}\left(Z^{-1}\cdot X_1(Z)\right) = -Z\cdot\frac{d}{dZ}\left(Z^{-1}\cdot\frac{1}{1-Z^{-1}}\right), \quad if \quad \left|\frac{1}{Z}\right| < 1$$

$$= -Z\cdot\frac{d}{dZ}\left((Z-1)^{-1}\right) = \frac{Z}{(Z-1)^2}$$

$$Therefore \quad X(Z) = \frac{Z}{(Z-1)^2} \quad R.O.C \quad |Z| > 1$$

(c)

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)\cdot Z^{-n} = \sum_{n=0,n:even}^{\infty} x(n)\cdot Z^{-n} = \sum_{m=0}^{\infty} x(2m)\cdot Z^{-2m} = \sum_{m=0}^{\infty}\alpha^{2m}\cdot Z^{-2m}$$

$$= \sum_{m=0}^{\infty}\left(\alpha^2 Z^{-2}\right)^m = \frac{1}{1-\alpha^2 Z^{-2}}, \quad if \quad \left|\frac{\alpha^2}{Z^2}\right| < 1$$

$$Therefore \quad X(Z) = \frac{1}{1-\alpha^2 Z^{-2}}, \quad R.O.C \quad |Z| > |\alpha|$$

Problem 4. For parts (a) and (b) of the following compute $x(n)$, the inverse z-transform, using any method you wish. For part (c) use the residue method.

a. $X(z) = \dfrac{z^2 - 2}{z - \frac{1}{3}}$, ROC corresponds to a right-sided sequence.

b. $X(z) = \dfrac{z^2}{z^2 - 5z + 6}$, ROC $= \{z : 2 < |z| < 3\}$.

c. $X(z) = \dfrac{z}{(z - \frac{1}{4})(z - \frac{1}{5})}$, ROC $= \left\{z : |z| > \dfrac{1}{4}\right\}$.

Evaluate your expression for $x(n)$ at $n = 0, 1, 2$ in each case.

(a) $X(Z) = \dfrac{Z^2 - 2}{Z - \frac{1}{3}} = Z + \dfrac{1}{3} - \dfrac{17}{9} \cdot Z^{-1} \cdot \dfrac{Z}{Z - \frac{1}{3}}$

Because x(n) is righted sequence, therefore the inverse Z transform of the above eq. can be easily derived.

$$x(n) = \delta(n+1) + \frac{1}{3} \cdot \delta(n) - \frac{17}{9} \cdot \left(\frac{1}{3}\right)^{n-1} \cdot u(n-1)$$

And $x(0) = \dfrac{1}{3}$, $x(1) = -\dfrac{17}{9}$, $x(2) = -\dfrac{17}{27}$

(b) $X(Z) = \dfrac{Z^2}{Z - 5Z + 6} = 1 + \dfrac{5Z - 6}{Z^2 - 5Z + 6} = 1 - 4Z^{-1} \cdot \dfrac{Z}{Z - 2} + 9Z^{-1} \cdot \dfrac{Z}{Z - 3}$

Because 2 < |Z| <3, therefore the 2nd term in the above eq. is right-sided seq. and the 3rd term of the above eq. is left-sided seq. The inverse Z transform of the above eq. can be easily derived.

$$x(n) = \delta(n) - 4 \cdot (2)^{n-1} \cdot u(n-1) - 9 \cdot (3)^{n-1} \cdot u(-n)$$

And $x(0) = -2$, $x(1) = -4$, $x(2) = -8$

(c) $X(Z) = \dfrac{Z}{(Z - \frac{1}{4})(Z - \frac{1}{5})}$

The inverse Z-transform x(n) can be calculated using residue method as following:

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)\cdot Z^{n-1}dz = \frac{1}{2\pi j}\oint_C \frac{Z}{(Z-\frac{1}{4})(Z-\frac{1}{5})}\cdot Z^{n-1}dz = \frac{1}{2\pi j}\oint_C \frac{Z^n}{(Z-\frac{1}{4})(Z-\frac{1}{5})}dz$$

$$= \sum_{Poles\quad inside\quad countour\quad C} residue(\frac{Z^n}{(Z-\frac{1}{4})(Z-\frac{1}{5})}) - \sum_{Poles\quad outside\quad countour\quad C} residue(\frac{Z^n}{(Z-\frac{1}{4})(Z-\frac{1}{5})})$$

where the countour C is chosen to be inside R.O.C.

The pole for X(Z) is 1/4 and 1/5. But because R.O.C. for X(Z) is |Z|>1/4, so all the poles are inside the contour and no pole is outside contour. Therefore x(n) can be derived by

$$x(n) = \sum_{Poles\quad inside\quad countour\quad C} residue(\frac{Z^n}{(Z-\frac{1}{4})(Z-\frac{1}{5})})$$

$$= \left[\frac{Z^n}{(Z-\frac{1}{4})(Z-\frac{1}{5})}\cdot(Z-\frac{1}{4})\right]_{Z=\frac{1}{4}}\cdot u(n) + \left[\frac{Z^n}{(Z-\frac{1}{4})(Z-\frac{1}{5})}\cdot(Z-\frac{1}{5})\right]_{Z=\frac{1}{5}}\cdot u(n)$$

$$= 20\cdot\left[\left(\frac{1}{4}\right)^n - \left(\frac{1}{5}\right)^n\right]\cdot u(n)$$

And $x(0) = 0, \quad x(1) = 1, \quad x(2) = \frac{9}{20}$

Problem 5. A certain sequence, $x(n)$, is a right-sided sequence such that $x(n) = 0$ for $n < 0$. The sequence has z-transform

$$X(z) = e^{z^{-1}}.$$

Find $x(4)$, i.e., evaluate the sequence $x(n)$ at $n = 4$. You may find the following result useful:

$$e^u = \sum_{k=0}^{\infty} \frac{u^k}{k!}.$$

Solution.

$$X(z) = 1 + z^{-1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \frac{z^{-4}}{4!} + \cdots$$

so

$$x(n) = \delta(n) + \delta(n-1) + \frac{\delta(n-2)}{2!} + \frac{\delta(n-3)}{3!} + \frac{\delta(n-4)}{4!} + \cdots$$

thus

$$x(4) = \frac{1}{24}.$$

Problem 6. A certain linear system has a response to a delayed unit step given by

$$s_k(n) = k\delta(n-k),$$

that is, $s_k(n)$ is the response of the linear system to the input $x(n) = u(n-k)$. Find the response of this system to the input $x(n) = \delta(n-k)$, where k is an arbitrary integer and determine whether or not the system is BIBO stable.

Solution.

Since

$$\delta(n) = u(n) - u(n-1)$$

then

$$\delta(n-k) = u(n-k) + u(n-k-1)$$

so

$$h_k(n) = k\delta(n-k) - (k+1)\delta(n-k-1).$$

For BIBO stability we check

$$\sum_{k=-\infty}^{\infty} |h_k(n)| < \infty.$$

We find

$$\sum_{k=-\infty}^{\infty} |k\delta(n-k) - (k+1)\delta(n-k-1)|$$
$$= |n| + |-(n-1+1)| = 2|n| \longrightarrow \infty$$

so the system is not BIBO stable.

Problem 7. Consider the following sequence:

$$y(n) = \sum_{j=0}^{n-1} \sum_{k=j+1}^{n} x(k).$$

Find a closed form expression for $Y(z)$ in terms of $X(z)$.
Solution.

Note that

$$y(n) = \sum_{k=1}^{n} kx(k)$$

then observe

$$y(n) - y(n-1) = nx(n)$$

so

$$Y(z) - z^{-1}Y(z) = -z\frac{d}{dz}X(z)$$

hence

$$Y(z) = -\frac{z^2}{z-1}\frac{d}{dz}X(z).$$

You could also obtain this solution by noting

$$y(n) = \sum_{k=1}^{n} kx(k) = \sum_{k=0}^{n} kx(k) = nx(n)u(n) * u(n)$$

and thus

$$Y(z) = -z\frac{d}{dz}X(z) \cdot \frac{z}{z-1} = -\frac{z^2}{z-1}\frac{d}{dz}X(z).$$

Problem 8. Define the *falling factorial polynomials* by $x[0] = 1$ and

$$[x]_n = x(x-1)(x-2)\cdots(x-n+1), \quad n = 1, 2, 3, \ldots.$$

The coefficient of x^r in $[x]_n$ is known as the Stirling number of the first kind and is denoted $s(n, r)$. Thus,

$$[x]_n = \sum_{r=0}^{n} s(n, r) x^r.$$

Now let $y(n) = [x]_n$ and let $x = \alpha,\ 0 < \alpha < 1$. Find the first-order linear differential equation that $Y(z)$ satisfies. For 5 points extra credit solve the differential equation for $Y(z)$.

Solution.

Note that

$$y(n+1) = (\alpha - n)y(n) + \delta(n+1)$$

so that

$$zY(z) - \alpha Y(z) - z\frac{d}{dz}Y(z) - z = 0$$

or

$$Y'(z) - \frac{z-\alpha}{z}Y(z) = -1.$$

To solve this differential equation note the integrating factor is

$$\exp\left(\int -\frac{z-\alpha}{z}dz\right) = e^{-z+\alpha\ln z} = e^{-z}z^{\alpha}.$$

Therefore,

$$Y(z)\left[e^{-z}z^{\alpha}\right] = -\int e^{-z}z^{\alpha}dz + c$$

or

$$Y(z) = -e^{z}z^{-\alpha}\int e^{-z}z^{\alpha}dz + c$$

where the constant c is chosen so that $Y(z)|_{z=\infty} = 1$ since $y(0) = 1$.