EE 113 Midterm Solutions

Spring 2006

Inst: Dr. C.W. Walker

Problem	Points	Score
1	9	
2	11	
3	15	
4	15	
5	16	
6	18	
7	16	
Total	100	

Problem 1. For parts (a), (b) and (c) determine whether or not the system is

i. linear

ii. time-invariant

iii. BIBO stable, i.e., bounded input-bounded output stable

a. $y(n) = x(n)$.

   Solution. linear, time-invariant, BIBO stable

b. $y(n) = nx(n)$.

   Solution. linear, not time-invariant, not BIBO stable

c. $y(n) = 2$.

   Solution. not linear, time-invariant, BIBO stable

Problem 2. Prove that for a causal LTI system, $h(n) = 0$ for $n < 0$.

***Proof*:**

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{n} x(k)h(n-k) + \sum_{k=n+1}^{\infty} x(k)h(n-k).$$

Since our system is causal the output at time n does not depend on the input for $k > n$ so the right hand expression above must be zero. However, since we have no control over the input we must have

$$h(n-k) = 0 \ \forall k \geq n+1 \Rightarrow h(n) = 0 \ \forall n < 0.$$

Problem 3. The impulse response for a causal LTI system is given. Determine whether or not the system is BIBO stable. Justify your answer.

a. $h(n) = \left(\frac{1}{2}\right)^n u(n)$.

   Solution. An LTI system is BIBO stable *iff* $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$.

$$\sum_{n=-\infty}^{\infty} \left| \left(\frac{1}{2}\right)^n u(n) \right| = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 2$$

   so this system is BIBO stable.

b. $h(n) = (-1)^n\ u(n)$.

Solution.

$$\sum_{n=-\infty}^{\infty} |(-1)^n\ u(n)| = \sum_{n=0}^{\infty} 1 = \infty$$

so this system is not BIBO stable.

c. $h(n) = \frac{1}{n}\ u(n-2)$.

Solution.

$$\sum_{n=-\infty}^{\infty} \left|\frac{1}{n}\ u(n-2)\right| = \sum_{n=2}^{\infty} \frac{1}{n} = \infty$$

so this system is not BIBO stable.

Problem 4. Consider the system described by the following difference equation:

$$y(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = x(n),$$

where,

$$x(n) = (1/4)^n\, u(n), \quad y(-1) = 1,\ y(-2) = 0.$$

a. Find a closed form expression for $y(n)$.

Solution.

$$Y^\dagger(z) - \frac{5}{6}z^{-1}\left[Y^\dagger(z) + y(-1)z\right] + \frac{1}{6}z^{-2}\left[Y(z) + y(-1)z + y(-2)z^2\right] = X^\dagger(z)$$

Here

$$X^\dagger(z) = \frac{z}{z-1/4}, \quad |z| > 1/4$$

We find

$$\frac{Y^\dagger(z)}{z} = \frac{3}{z-1/4} + \frac{15/2}{z-1/2} - \frac{26/3}{z-1/3}$$

so

$$y(n) = \left[3 \cdot \left(\frac{1}{4}\right)^n + \frac{15}{2} \cdot \left(\frac{1}{2}\right)^n - \frac{26}{3} \cdot \left(\frac{1}{3}\right)^n\right] u(n)$$

b. Evaluate your $y(n)$ at $n = 30$.

Solution.

$$y(30) = 6.98 \times 10^{-9}$$

Problem 5. Compute $X(z)$, the forward z-transform, (if it exists) for each of the following. Remember to specify the region of convergence in each case. If the forward z-transform does not exist, explain why.

a. $x(n) = \left(\frac{1}{3}\right)^n u(n-1)$.

Solution.

$$X(z) = \frac{1}{3} \cdot \frac{1}{z - 1/3}, \quad |z| > 1/3$$

b. $x(n) = 2^n u(n-2)$.

Solution.

$$X(z) = 4 \cdot \frac{z^{-1}}{z-2}, \quad |z| > 2$$

c. $x(n) = 2^{n^2} u(n)$.

Solution. The z-transform does not exist.

d.

$$y(n) = \begin{cases} \alpha^n u(n), & n \text{ is a multiple of } 4, \\ 0, & \text{elsewhere,} \end{cases}$$

and

$$x(n) = y(4n).$$

Note that 0 is a multiple of 4.

Solution. Observe $x(n) = \alpha^{4n} u(n)$. So

$$X(z) = \frac{z}{z - \alpha^4}, \quad |z| > |\alpha|^4$$

Problem 6. For part (a) and (b) of the following compute $x(n)$, the inverse z-transform, using any method you wish. For part (c) use the residue formula. Evaluate your expression for $x(n)$ at $n = 0, 1, 2$ in each case.

a. $X(z) = \dfrac{z^2 - 1}{z - \frac{1}{2}}$, ROC corresponds to a right-sided sequence.

Solution.

$$X(z) = z + \frac{1}{2} - \frac{3/4}{z - 1/2}$$

so

$$x(n) = \delta(n+1) + \frac{1}{2}\delta(n) - \frac{3}{4}\left(\frac{1}{2}\right)^{n-1} u(n).$$

$$x(0) = \frac{1}{2}, \quad x(1) = -\frac{3}{4}, \quad x(2) = -\frac{3}{8}.$$

b. $X(z) = \dfrac{z^2}{z^2 - 7z + 10}, \quad \text{ROC} = \{z : 2 < |z| < 5\}.$

Solution.

$$X(z) = \frac{5}{3}\frac{z}{z-5} - \frac{2}{3}\frac{z}{z-2}$$

so

$$x(n) = -\frac{5}{3}(5)^n u(-n-1) - \frac{2}{3}(2)^n u(n).$$

$$x(0) = -\frac{2}{3}, \quad x(1) = -\frac{4}{3}, \quad x(2) = -\frac{8}{3}.$$

c. $X(z) = \dfrac{z}{(z-\frac{1}{2})(z-\frac{1}{4})}, \quad \text{ROC} = \left\{z : |z| > \frac{1}{2}\right\}.$

$$\begin{aligned} x(n) = & \sum_{\substack{\text{all poles} \\ \text{inside C}}} \text{Res } X(z)z^{n-1}, \quad m \geq 0 \\ & - \sum_{\substack{\text{all poles} \\ \text{outside C}}} \text{Res } X(z)z^{n-1}, \quad m < 0 \end{aligned}$$

where, m is the least degree of the numerator polynomial of $X(z)z^{n-1}$.

$$\begin{aligned} = & \sum_{\substack{\text{all poles} \\ \text{inside C}}} \text{Res } \frac{z}{(z-\frac{1}{2})(z-\frac{1}{4})} z^{n-1}, \quad m \geq 0 \\ & - \sum_{\substack{\text{all poles} \\ \text{outside C}}} \text{Res } \frac{z}{(z-\frac{1}{2})(z-\frac{1}{4})} z^{n-1}, \quad m < 0. \end{aligned}$$

The numerator is

$$z^n \Rightarrow m = n$$

$$m \geq 0 \Rightarrow n \geq 0$$

$$m < 0 \Rightarrow n \leq -1.$$

So,

$$x(n) = \frac{z^n}{(z-\frac{1}{2})(z-\frac{1}{4})}\left(z-\frac{1}{2}\right)\Big|_{z=1/2} u(n)$$

$$+\frac{z^n}{(z-\frac{1}{2})(z-\frac{1}{4})}\left(z-\frac{1}{4}\right)\Big|_{z=1/4} u(n)$$

or

$$x(n) = \left[4\left(\frac{1}{2}\right)^n - 4\left(\frac{1}{4}\right)^n\right] u(n).$$

$$x(0) = 0, \quad x(1) = 1, \quad x(2) = \frac{3}{4}.$$

Problem 7. A certain sequence, $x(n)$, is a right-sided sequence such that $x(n) = 0$ for $n < 0$. The sequence has z-transform

$$X(z) = e^{e^{z^{-1}}}.$$

Find $x(4)$, i.e., evaluate the sequence $x(n)$ at $n = 4$. You may find the following result useful:

$$e^u = \sum_{k=0}^{\infty} \frac{u^k}{k!}.$$

Solution 1.

$$\frac{d}{dz}X(z) = \frac{d}{dz}\left(e^{e^{z^{-1}}}\right) = \frac{d}{dz}\left(e^{z^{-1}}\right) \cdot \left(e^{e^{z^{-1}}}\right) = -z^{-2}\left(e^{z^{-1}}\right)\left(e^{e^{z^{-1}}}\right)$$

$$-z\frac{d}{dz}X(z) = z^{-1}e^{z^{-1}}X(z)$$

Now

$$e^{z^{-1}} = 1 + z^{-1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \cdots$$

$$z^{-1}e^{z^{-1}} = z^{-1} + z^{-2} + \frac{z^{-3}}{2!} + \frac{z^{-4}}{3!} + \cdots$$

so

$$nx(n) \Longleftrightarrow \left(z^{-1} + z^{-2} + \frac{z^{-3}}{2!} + \frac{z^{-4}}{3!} + \cdots\right) X(z)$$

thus

$$nx(n) = x(n-1) + x(n-2) + \frac{x(n-3)}{2!} + \frac{x(n-4)}{3!} + \cdots$$

If we evaluate $X(z)$ at $z = \infty$ we find $x(0) = e$ and thus

$$x(1) = x(0) = e$$

$$x(2) = \frac{x(1) + x(0)}{2} = e$$

$$x(3) = \frac{x(2) + x(1) + \frac{x(0)}{2}}{3} = \frac{5}{6}e$$

$$x(4) = \frac{x(3) + x(2) + \frac{x(1)}{2} + \frac{x(0)}{6}}{4} = \frac{5}{8}e$$

Solution 2.

$$X(z) = \sum_{k=0}^{\infty} \frac{\left(e^{z^{-1}}\right)^k}{k!} = \sum_{k=0}^{\infty} \frac{e^{k/z}}{k!} = \sum_{k=0}^{\infty} \frac{\sum_{n=0}^{\infty} \frac{(k/z)^n}{n!}}{k!} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(k/z)^n}{k!n!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(k/z)^n}{k!n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{k^n}{k!n!} z^{-n}$$

so

$$x(n) = \sum_{k=0}^{\infty} \frac{k^n}{k!n!}$$

and

$$x(4) = \sum_{k=0}^{\infty} \frac{k^4}{k!4!}$$

Let

$$f(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$xf'(x) = \sum_{k=0}^{\infty} \frac{kx^k}{k!} = xe^x$$

$$x\left(f'(x)\right)' = \sum_{k=0}^{\infty} \frac{k^2x^k}{k!} = (x^2 + x)e^x$$

$$x\left(\left(f'(x)\right)'\right)' = \sum_{k=0}^{\infty} \frac{k^3 x^k}{k!} = (x^3 + 3x^2 + x)e^x$$

$$x\left(\left(\left(f'(x)\right)'\right)'\right)' = \sum_{k=0}^{\infty} \frac{k^4 x^k}{k!} = (x^4 + 6x^3 + 7x^2 + x)e^x$$

Let $x = 1$ to get

$$\sum_{k=0}^{\infty} \frac{k^4}{k!} = 15e$$

and thus

$$x(4) = \sum_{k=0}^{\infty} \frac{k^4}{k!4!} = \frac{15e}{24} = \frac{5}{8}e$$