Problem 1. For parts (a), (b) and (c) determine whether or not the system is

i. linear

ii. time-invariant

iii. BIBO stable, i.e., bounded input-bounded output stable

a. $y(n) = x(n) + x(n-1) + 1.$

b. $y(n) = \cos\left[x(n)\right].$

c. $y(n) = [x(n)]^n.$

Note: You do not need to show a lot of work on this problem if you can quickly recognize the answer.

Solution.

a. $y(n) = x(n) + x(n-1) + 1$. Clearly Non-linear and Time-invariant. Also BIBO stable: $|x(n)| < M\ \forall\ n \Rightarrow |y(n)| < 2M + 1\ \forall\ n$.

b. $y(n) = \cos\left[x(n)\right]$. Clearly Non-linear and Time-invariant. BIBO stable because $|y(n)| = |\cos[x(n)]| \leq 1\ \forall\ n$.

c. $y(n) = [x(n)]^n$. Clearly Non-linear.
Not Time-invariant because if $y(n) = \mathcal{T}[x(n)] = [x(n)]^n$, then
$\mathcal{T}[x(n-k)] = [x(n-k)]^n \neq y(n-k) = [x(n-k)]^{n-k}$
Not BIBO stable: if $x(n) = 2\ \forall\ n$ (bounded), then $y(n) = 2^n$ is unbounded as $n \to \infty$.

Problem 2. Consider the system described by the following difference equation:

$$y(n) - \frac{11}{10}y(n-1) + \frac{1}{10}y(n-2) = x(n),$$

where,

$$x(n) = (1/3)^n\, u(n), \quad y(-1) = 1,\ y(-2) = 0.$$

Find a closed form expression for $y(n)$.

Solution. Method I: Time-domain

$$y(n) = y_h(n) + y_p(n)$$

The characteristic equation is $\lambda^2 - \frac{11}{10}\lambda - \frac{1}{10} = 0$. The roots are $\lambda = \frac{1}{10}, 1$. Therefore

$$y_h(n) = C_1 \left(\frac{1}{10}\right)^n + C_2(1)^n$$

$$x(n) = (1/3)^n u(n) \Rightarrow y_p(n) = K(1/3)^n u(n)$$

For $n \geq 2$,

$$\begin{aligned}
y_p(n) - \frac{11}{10}y_p(n-1) + \frac{1}{10}y_p(n-2) &= x(n) \\
K\left(\frac{1}{3}\right)^n - \frac{11}{10}K\left(\frac{1}{3}\right)^{n-1} + \frac{1}{10}K\left(\frac{1}{3}\right)^{n-2} &= \left(\frac{1}{3}\right)^n \\
K - \frac{11}{10}K(3) + \frac{1}{10}K(9) &= 1
\end{aligned}$$

$$\Rightarrow K = -\frac{5}{7} = 0.7143$$

$$y(n) = C_1\left(\frac{1}{10}\right)^n + C_2 - \frac{5}{7}\left(\frac{1}{3}\right)^n$$

To get C_1 and C_2, we first find $y(0)$ and $y(1)$ by propagating the difference equation.

$$\begin{aligned}
y(0) &= \frac{11}{10}y(-1) - \frac{1}{10}y(-2) + x(0) = \frac{11}{10} - 0 + 1 = \frac{21}{10} \\
y(1) &= \frac{11}{10}y(0) - \frac{1}{10}y(-1) + x(1) = \frac{11}{10}\cdot\frac{21}{10} - \frac{1}{10} + \frac{1}{3} = \frac{763}{300}
\end{aligned}$$

Using the expression for $y(n)$ we get the equations:

$$C_1 + C_2 + \left(-\frac{5}{7}\right) = \frac{21}{10}$$

$$C_1\left(\frac{1}{10}\right) + C_2 + \left(-\frac{5}{7}\right)\left(\frac{1}{3}\right) = \frac{763}{300}$$

Solving the above two equations we get $C_1 = \frac{23}{630} = 0.0365,\ C_2 = \frac{25}{9} = 2.7778$. Therefore

$$y(n) = \left[(0.0365)\left(\frac{1}{10}\right)^n + (2.7778) - 0.7143\left(\frac{1}{3}\right)^n\right]u(n)$$

Method II: Z-Transform

Taking the one sided Z-Transform of the difference equation and using the initial conditions, we get

$$Y^+(z) - \frac{11}{10}(1 + z^{-1}Y^+(z)) + \frac{1}{10}(z^{-1} + z^{-2}Y^+(z)) = X^+(z) = \frac{1}{1 - \frac{1}{3}z^{-1}}$$

$$\left(1 - \frac{11}{10}z^{-1} + \frac{1}{10}z^{-2}\right)Y^+(z) = \frac{1}{1 - \frac{1}{3}z^{-1}} + \left(\frac{11}{10} - \frac{1}{10}z^{-1}\right)$$

$$\begin{aligned} Y^+(z) &= \frac{21 - \frac{14}{3}z^{-1} + \frac{1}{3}z^{-2}}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - \frac{1}{10}z^{-1}\right)\left(1 - z^{-1}\right)} \\ &= \frac{(23/630)}{1 - \frac{1}{10}z^{-1}} + \frac{(25/9)}{1 - z^{-1}} + \frac{(-5/7)}{1 - \frac{1}{3}z^{-1}} \end{aligned}$$

Therefore as before:

$$y(n) = \left[(0.0365)\left(\frac{1}{10}\right)^n + (2.7778) - 0.7143\left(\frac{1}{3}\right)^n\right]u(n)$$

Problem 3. Compute $X(z)$, the forward z-transform, (if it exists) for each of the following. Remember to specify the region of convergence in each case. If the forward z-transform does not exist, explain why.

a. $x(n) = \left(\frac{1}{2}\right)^n u(n-3)$.

b. $x(n) = 2^n u(n-1)$.

c. $x(n) = \left(\alpha^n + \alpha^{-n}\right)u(n)$.

d. $x(n) = 2^{n^2} u(n)$.

e.

$$x(n) = \begin{cases} \alpha^n u(n), & n \text{ is a multiple of 2,} \\ 0, & \text{elsewhere.} \end{cases}$$

Solution.

a. $x(n) = \left(\frac{1}{2}\right)^n u(n-3) = \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{n-3} u(n-3)$

$$X(z) = \left(\frac{1}{2}\right)^3 \cdot \frac{z^{-3}}{1-\frac{1}{2}z^{-1}} = \frac{1}{4z^2(2z-1)}, \text{ ROC: } |z| > \frac{1}{2}$$

b. $x(n) = 2^n u(n-1) = 2 \cdot 2^{n-1} u(n-1)$

$$X(z) = \frac{2z^{-1}}{1-2z^{-1}} = \frac{2}{z-2} \text{ , ROC : } |z| > 2$$

c. $x(n) = \alpha^n u(n) + \alpha^{-n} u(n)$

$$\mathcal{Z}[\alpha^n u(n)] = \frac{1}{1-\alpha z^{-1}}, \text{ ROC : } |z| > |\alpha|$$

$$\mathcal{Z}[\alpha^{-n} u(n)] = \frac{1}{1-\frac{1}{\alpha}z^{-1}}, \text{ ROC : } |z| > \frac{1}{|\alpha|}$$

Therefore

$$X(z) = \frac{1}{1-\alpha z^{-1}} + \frac{1}{1-\frac{1}{\alpha}z^{-1}}, \text{ ROC : } |z| > \max\{|\alpha|, \frac{1}{|\alpha|}\}$$

d. $x(n) = 2^{n^2} u(n)$.

$$X(z) = \sum_{n=0}^{\infty} 2^{n^2} z^{-n} = \sum_{n=0}^{\infty} (2^n z^{-1})^n$$

Since $2^n z^{-1}$ is an unbounded sequence for any z, it is clear that the Z-Transform of $x(n)$ does not exist.

e.

$$\begin{aligned} X(z) &= \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \alpha^n z^{-n} = \sum_{m=0}^{\infty} \alpha^{2m} z^{-2m} \text{ (substituting } n = 2m) \\ &= \frac{1}{1-\alpha^2 z^{-2}} \text{ , ROC : } |z| > |\alpha| \end{aligned}$$

Problem 4. Find $x(n)$, the inverse z-transform, using any method you wish.

a. $X(z) = \dfrac{z^2-4}{z-\frac{1}{3}}$, ROC corresponds to a right-sided sequence.

b. $X(z) = \dfrac{z^2}{z^2-5z+6}$, ROC $= \{z : 2 < |z| < 3\}$.

c. $X(z) = \dfrac{z}{(z-\frac{1}{2})(z-\frac{1}{3})}$, ROC $= \left\{z : |z| > \dfrac{1}{2}\right\}$.

Evaluate your expression for $x(n)$ at $n = 0, 1, 2$ in each case.

Solution.

a. $X(z) = \dfrac{z^2-4}{z-\frac{1}{3}} = \dfrac{z^2-\frac{1}{9}-\frac{35}{9}}{z-\frac{1}{3}} = z+\dfrac{1}{3}+\dfrac{(-35/9)}{z-\frac{1}{3}} = z+\dfrac{1}{3}+\left(\dfrac{-35}{9}\right) z^{-1}\dfrac{z}{z-\frac{1}{3}}$

$$x(n) = \delta(n+1) + \frac{1}{3}\delta(n) + \left(-\frac{35}{9}\right)\left(\frac{1}{3}\right)^{n-1} u(n-1)$$

$x(0) = \dfrac{1}{3}, x(1) = -\dfrac{35}{9}, x(2) = -\dfrac{35}{27}$

b. $X(z) = \dfrac{z^2}{z^2-5z+6} = \dfrac{z^2}{(z-2)(z-3)} = \dfrac{1}{(1-2z^{-1})(1-3z^{-1})}$

$$\begin{aligned} X(z) &= \frac{(-2)}{1-2z^{-1}} + \frac{3}{1-3z^{-1}} \\ &= \frac{(-2)}{1-2z^{-1}} + \frac{z}{1-\frac{1}{3}z}, \text{ ROC} = \{z : 2 < |z| < 3\} \\ x(n) &= -2(2)^n u(n) - \left(\frac{1}{3}\right)^{-n-1} u(-n-1) \end{aligned}$$

$$x(0) = -2,\ x(1) = -4,\ x(3) = -8$$

c.

$$\begin{aligned}
X(z) &= \frac{z}{(z-\frac{1}{2})(z-\frac{1}{3})} \\
&= \frac{z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)} \\
&= \frac{6}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-\frac{1}{3}z^{-1}}, \quad \text{ROC} = \left\{z : |z| > \frac{1}{2}\right\} \\
x(n) &= \left[6\left(\frac{1}{2}\right)^n - 6\left(\frac{1}{3}\right)^n\right]u(n)
\end{aligned}$$

$$x(0) = 0,\ x(1) = 1,\ x(2) = \frac{5}{6}$$

Problem 5. A proposed form for a z-transform of a signal, $x(n)$, is given as

$$X(z) = \frac{z}{z-\alpha} + \frac{z}{z-\beta}$$

subject to the following constraints:

1. $|\alpha| \neq |\beta|$.
2. $\alpha + \beta = 1$.
3. $x(n)x(-n) = -\frac{1}{2}$ when $n = 1$.

Given the constraints, if there are any valid regions of convergence for $X(z)$, find them. If none exist, then show why not. You may assume without loss of generality that $|\alpha| > |\beta|$.

Solution. Assuming $|\alpha| > |\beta|$, there are three cases for the ROC.

Case I: $|z| > |\alpha|$
Then $x(n) = (\alpha^n + \beta^n)u(n)$, and so $x(-1) = 0$ and constraint (3) is not possible.

Case II: $|z| < |\beta|$
Then $x(n) = -(\alpha^n + \beta^n)u(-n-1)$, and so $x(1) = 0$ and constraint (3) is again not possible.
Case III: $|\beta| < |z| < |\alpha|$
Then $x(n) = -\alpha^n u(-n-1) + \beta^n u(n)$, and so
$x(1) = \beta,\ x(-1) = -\alpha$, and constraint (3) becomes

$$-\frac{\beta}{\alpha} = -\frac{1}{2} \Rightarrow \alpha = 2\beta$$

Taken together with Constraint (2), $\alpha + \beta = 1$, we get $\alpha = \dfrac{2}{3},\ \beta = \dfrac{1}{3}$
This gives us the valid ROC: $\dfrac{1}{3} < |z| < \dfrac{2}{3}$.

Problem 6. Consider the following sequence:

$$y(n) = \sum_{k=0}^{n} x(k).$$

If $X(z)$ has the value of 1 at $z = \frac{1}{2}$ find the value of $Y(z)$ at $z = \frac{1}{2}$.

Solution. If $x(n)$ is assumed to be a causal sequence, then

$$y(n) = \sum_{k=0}^{n} x(k) = x(n) * u(n)$$

$$\Rightarrow Y(z) = X(z) \cdot \frac{1}{1 - z^{-1}} \quad \text{wrong!}$$

since the ROC of $\mathcal{Z}[u(n)] = \dfrac{1}{1 - z^{-1}}$ is $|z| > 1$, the ROC of $Y(z)$ is the intersection of the ROC of $X(z)$ and $|z| > 1$, and so $Y(z)$ cannot exist at $z = \frac{1}{2}$.

However, if $x(n)$ is an anti-causal sequence ($= 0$ for $n > 0$), then $y(n) = \sum_{k=0}^{n} x(k)$ will also be anti-causal, and

$$y(n) = \sum_{k=0}^{n} x(k) = x(n) * u(-n)$$

$$\Rightarrow Y(z) = X(z) \cdot \frac{1}{1 - z}$$

which is okay since the ROC of $\mathcal{Z}[u(-n)] = \dfrac{1}{1-z}$ is $|z| < 1$, and so $Y(z)$ will exist for $z = \frac{1}{2}$ if $X(z)$ does.

$$\Rightarrow Y\left(\frac{1}{2}\right) = X\left(\frac{1}{2}\right) \cdot \frac{1}{1-\frac{1}{2}} = 2$$

Problem 7. Consider the famous *Fibonacci sequence*, $F(n)$,

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13, \ldots$$

where 0 is considered the origin, i.e., the sequence is 0 when $n = 0$. Thus, $F(0) = 0,\ F(1) = 1,\ F(2) = 1,\ F(3) = 2$, etc.

Find a closed form expression for the nth term of the sequence $F(n)$.

Solution. This is from HW#2 with a shift of 1 to the right.

$$F(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \quad n \geq 1$$

Problem 8. Suppose we receive the analog signal

$$r_a(t) = A\cos(2\pi F t + \theta)$$

and sample it at 100 Hz to get the digital signal, $r(n)$.

Here the amplitude A is a constant but we do *not* know its value and we do *not* know the phase value θ. But we do know that the analog frequency is $F = 20$ Hz.

Explain clearly and in detail (using some math) an algorithm to use to estimate the amplitude A using $r(n)$. State under what circumstances your algorithm will give the exact answer for A.

Notes: We are not considering any effects due to A/D conversion here.

Solution.

$$r_a(t) = A\cos(2\pi 20t + \theta) = A\cos(40\pi t + \theta)$$

$$r(n) = r_a\left(\frac{n}{100}\right) == A\cos\left(\frac{2n\pi}{5} + \theta\right)$$

The algorithm is as described in problem 5 of HW 1. Briefly, we pick a positive integer N and perform the following operations to obtain $\hat{A}$ the estimate for A.

$$\begin{aligned}
z_1 &= \frac{2}{N}\sum_{n=0}^{N-1} r(n)\cos\left(\frac{2n\pi}{5}\right) \\
z_2 &= \frac{2}{N}\sum_{n=0}^{N-1} r(n)\sin\left(\frac{2n\pi}{5}\right) \\
\hat{A} &= \sqrt{z_1^2 + z_2^2}
\end{aligned}$$

As $N \to \infty$, $\hat{A} \to A$.

To find the values of N for which $\hat{A} = A$, note that

$$\begin{aligned}
z_1 &= \frac{A}{N}\sum_{n=0}^{N-1} 2\cos\left(\frac{2n\pi}{5} + \theta\right)\cos\left(\frac{2n\pi}{5}\right) \\
&= \frac{A}{N}\sum_{n=0}^{N-1}\left[\cos(\theta) - \cos\left(\frac{4n\pi}{5} + \theta\right)\right] \\
&= A\cos(\theta) + \frac{A}{N}\sum_{n=0}^{N-1}\cos\left(\frac{4n\pi}{5} + \theta\right) \\
\text{Similarly, } z_2 &= -A\sin(\theta) + \frac{A}{N}\sum_{n=0}^{N-1}\sin\left(\frac{4n\pi}{5} + \theta\right)
\end{aligned}$$

Since $\cos\left(\frac{4n\pi}{5} + \theta\right)$ and $\sin\left(\frac{4n\pi}{5} + \theta\right)$ are periodic with period 5,

$$\sum_{n=0}^{N-1}\cos\left(\frac{4n\pi}{5} + \theta\right) = 0$$

$$\text{and } \sum_{n=0}^{N-1}\cos\left(\frac{4n\pi}{5} + \theta\right) = 0$$

when N is a positive multiple of 5, and it is clear that $\hat{A} = A$ for these values of N.