1. Consider the following two systems

$$y_1(n) = \frac{1}{9}\sum_{k=n-2}^{n} x(k),$$
$$y_2(n) = \begin{cases} x\left(\frac{n}{5}\right) & n = 5k, k \text{ integer} \\ 0 & n \neq 5k, k \text{ integer}. \end{cases}$$

(a) Determine whether the systems are linear, time-invariant, relaxed, BIBO stable, and causal. Justify your answer to receive full credit.

Solution:

Properties	$y_1(n)$	$y_2(n)$
Relaxed	Yes	No
Linear	Yes	Yes
Time-Invariant	Yes	No
BIBO Stable	Yes	Yes
Causal	Yes	No

- System $y_1(n)$:
  - $y_1(n) = \frac{1}{9}\sum_{k=n-2}^{n} x(k) = \frac{1}{9}[x(n-2) + x(n-1) + x(n)]$ can be considered as a relaxed constant coefficient difference equation. It is relaxed, linear, time-invariant, BIBO stable, and causal.
- System $y_2(n)$:
  - Not relaxed: Counter example: if the first non-zero sample of $x(n)$ is at $n = -1$, then $y_2(-5)$ will be non-zero, which is before -1.
  - Linear: Let two outputs be

$$y_2^a(n) = \begin{cases} x_a\left(\frac{n}{5}\right) & n = 5k, k \text{ integer} \\ 0 & n \neq 5k, k \text{ integer}. \end{cases},$$

and

$$y_2^b(n) = \begin{cases} x_b\left(\frac{n}{5}\right) & n = 5k, k \text{ integer} \\ 0 & n \neq 5k, k \text{ integer}. \end{cases},$$

for an input $x_a(n)$ and $x_b(n)$, respectively. The output $y(n)$ for an input $Ax_a(n)+Bx_b(n)$ with some constants A and B is given by
i) if $n = 5k$, k integer,

$$\begin{aligned} y(n) &= Ax_a\left(\frac{n}{5}\right) + Bx_b\left(\frac{n}{5}\right) \\ &= Ay_2^a(n) + By_2^b(n). \end{aligned}$$

ii) if $n \neq 5k$, k integer,

$$y(n) = 0.$$

Based on i) and ii), $y(n)$ for an input $Ax_a(n) + Bx_b(n)$ is

$$y(n) = Ay_2^a(n) + By_2^b(n),$$

which implies that system $y_2(n)$ is linear.

- BIBO Stable: For a bounded sequence $|x(n)| \leq M < \infty$ with a finite positive number M,

$$|y_2(n)| = \begin{cases} |x\left(\frac{n}{5}\right)| \leq M < \infty & n = 5k, k \text{ integer} \\ 0 < \infty & n \neq 5k, k \text{ integer.} \end{cases},$$

implying that $|y_2(n)|$ is bounded output.

- Not Time-Invariant: Consider inputs $\delta(n)$ and $\delta(n-1)$. The outputs are $\delta(n)$ and $\delta(n-5)$. This shows that $y_2(n)$ is not time-invariant.
- Not causal: For example, $y_2(-5)$ depends on $x(-1)$, which is on the future of $y_2(-5)$.

(b) Given $x(n) = nu(n)$, compute and plot $y_1(n)$ and $y_2(n)$ for $0 \leq n \leq 5$.

Solution:

n	0	1	2	3	4	5
$y_1(n)$	0	1/9	3/9=1/3	6/9 = 2/3	9/9 = 1	12/9 = 4/3
$y_2(n)$	0	0	0	0	0	1

The corresponding plots are shown in Figure 1.

(c) Find the z-transform of $y_1(n)$ and $y_2(n)$ in terms of the z-transform of $x(n)$.

Solution:
z-transform for $y_1(n) = \frac{1}{9}\sum_{k=n-2}^{n} x(k) = \frac{1}{9}[x(n-2) + x(n-1) + x(n)]$ gives

$$\begin{aligned} Y_1(z) &= \frac{1}{9}[z^{-2}X(z) + z^{-1}X(z) + X(z)] \\ &= \frac{z^{-2} + z^{-1} + 1}{9}X(z). \end{aligned}$$

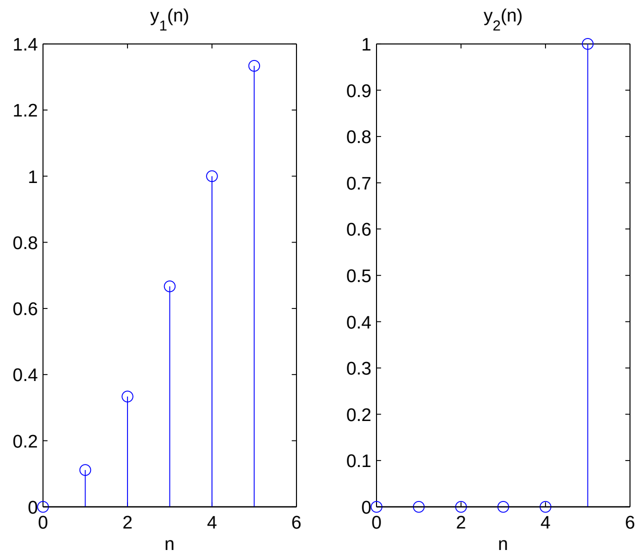
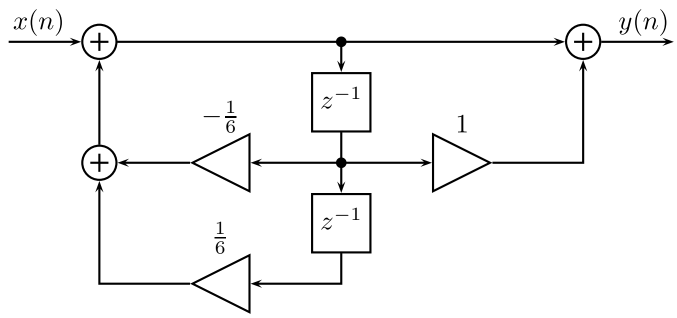


Figure 1: Plot for $y_1(n)$ and $y_2(n)$.

z-transform for $y_2(n)$ is given by

$$\begin{aligned} Y_2(z) &= \sum_{n=-\infty}^{\infty} y_2(n) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{5}\right) z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x(k) z^{-5k} \\ &= \sum_{k=-\infty}^{\infty} x(k) (z^5)^{-k} \\ &= X(z^5). \end{aligned}$$

2. A causal system is described by the following block diagram.



(a) Determine the constant coefficient difference equation that describes the system, and find its impulse response.

Solution:
Interchanging the order of the two systems in cascade, we get

$$y(n) = -\frac{1}{6}y(n-1) + \frac{1}{6}y(n-2) + x(n) + x(n-1).$$

The impulse response can be found by computing the zero-state response to $x(n) = \delta(n)$. We first find the homogeneous solution:

$$\lambda^2 + \frac{1}{6}\lambda - \frac{1}{6} = 0 \quad \Rightarrow \quad \lambda_1 = -\frac{1}{2}, \quad \lambda_2 = \frac{1}{3}$$

and hence,

$$y_h(n) = C_1\left(-\frac{1}{2}\right)^n + C_2\left(\frac{1}{3}\right)^n, \quad \text{for all } n.$$

Letting $x(n) = \delta(n)$ and assuming that the system is relaxed, we get $y(0) = x(0) = 1$ and $y(1) = -\frac{1}{6}y(0) + x(0) = \frac{5}{6}$. From the homogeneous solution we get

$$\begin{aligned} y(0) &= C_1 + C_2 = 1 \\ y(1) &= -\frac{1}{2}C_1 + \frac{1}{3}C_2 = \frac{5}{6} \end{aligned}$$

and solving for C_1 and C_2 yields $C_1 = -\frac{3}{5}$ and $C_2 = \frac{8}{5}$, and consequently,

$$h(n) = \left[-\frac{3}{5}\left(-\frac{1}{2}\right)^n + \frac{8}{5}\left(\frac{1}{3}\right)^n\right]u(n).$$

(b) Given $x(n) = 2^n u(-n)$, find the output of the system.

Solution:

Using the impulse response in (a), the output is

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{+\infty} 2^{n-k} u(-(n-k)) \left[-\frac{3}{5}\left(-\frac{1}{2}\right)^k + \frac{8}{5}\left(\frac{1}{3}\right)^k \right] u(k) \tag{1}$$

If $n < 0$,

$$\begin{aligned} y(n) &= \sum_{k=0}^{+\infty} 2^{n-k} \left[-\frac{3}{5}\left(-\frac{1}{2}\right)^k + \frac{8}{5}\left(\frac{1}{3}\right)^k \right] &(2)\\ &= 2^n \sum_{k=0}^{+\infty} \left[-\frac{3}{5}\left(-\frac{1}{4}\right)^k + \frac{8}{5}\left(\frac{1}{6}\right)^k \right] &(3)\\ &= -\frac{36}{25} 2^n &(4) \end{aligned}$$

If $n \geq 0$,

$$\begin{aligned} y(n) &= \sum_{k=n}^{+\infty} 2^{n-k} \left[-\frac{3}{5}\left(-\frac{1}{2}\right)^k + \frac{8}{5}\left(\frac{1}{3}\right)^k \right] &(5)\\ &= 2^n \sum_{k=n}^{+\infty} \left[-\frac{3}{5}\left(-\frac{1}{4}\right)^k + \frac{8}{5}\left(\frac{1}{6}\right)^k \right] &(6)\\ &= 2^n \left[-\frac{12}{25}\left(-\frac{1}{4}\right)^n + \frac{48}{25}\left(\frac{1}{6}\right)^n \right] &(7)\\ &= -\frac{12}{25}\left(-\frac{1}{2}\right)^n + \frac{48}{25}\left(\frac{1}{3}\right)^n &(8) \end{aligned}$$

Therefore,

$$y(n) = \begin{cases} -\frac{36}{25} 2^n, & \text{if } n < 0 \\ -\frac{12}{25}\left(-\frac{1}{2}\right)^n + \frac{48}{25}\left(\frac{1}{3}\right)^n, & \text{if } n \geq 0 \end{cases}$$

3. The difference equation of a relaxed system is:

$$y(n) + 0.5y(n-1) - 0.36y(n-2) = x(n)$$

(a) Find a closed form for the impulse response $h(n)$ (i.e., the zero state system output when the input is an impulse).

Solution:
By setting $x(n) = \delta(n)$, we have

$$h(n) + 0.5h(n-1) - 0.36h(n-2) = \delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$

Hence, for $n \geq 1$, $h(n)$ can be found by solving homogeneous difference equation

$$h(n) + 0.5h(n-1) - 0.36h(n-2) = 0. \tag{9}$$

The characteristic polynomial for (9) can be solved by

$$\left(\lambda + \frac{9}{10}\right)\left(\lambda - \frac{2}{5}\right) = 0,$$

thus, the modes are

$$\lambda_1 = -\frac{9}{10}, \quad \lambda_2 = \frac{2}{5}.$$

Hence,

$$h(n) = C_1\left(-\frac{9}{10}\right)^n + C_2\left(\frac{2}{5}\right)^n, \quad n \geq 0.$$

Since the system is relaxed, $y(-1) = h(-1) = 0$ and $h(0) = \delta(0) = 1$, which gives

$$C_1 = \frac{9}{13}, \quad C_2 = \frac{4}{13}.$$

Therefore,

$$h(n) = \left[\frac{9}{13}\left(-\frac{9}{10}\right)^n + \frac{4}{13}\left(\frac{2}{5}\right)^n\right]u(n).$$

(b) If the input to the system is $x(n) = 2\delta(n) + \delta(n-2)$, what is the output?

Solution:
The output $y(n)$ of the system can be expressed as the convolution of $x(n)$ and $h(n)$, i.e.,

$$\begin{aligned} y(n) &= x(n) * h(n) \\ &= [2\delta(n) + \delta(n-2)] * h(n) \\ &= 2h(n) + h(n-2). \end{aligned}$$

Hence, using $h(n)$ in part (a) gives us
i) $n \geq 2$,

$$\begin{aligned}
y(n) &= 2h(n) + h(n-2) \\
&= 2\left[\frac{9}{13}\left(-\frac{9}{10}\right)^n + \frac{4}{13}\left(\frac{2}{5}\right)^n\right]u(n) + \left[\frac{9}{13}\left(-\frac{9}{10}\right)^{n-2} + \frac{4}{13}\left(\frac{2}{5}\right)^{n-2}\right]u(n-2) \\
&= \left[2\cdot\frac{9}{13} + \frac{9}{13}\cdot\left(\frac{9}{10}\right)^{-2}\right]\left(-\frac{9}{10}\right)^n + \left[2\cdot\frac{4}{13} + \frac{4}{13}\cdot\left(\frac{2}{5}\right)^{-2}\right]\left(\frac{2}{5}\right)^n \\
&= \frac{262}{117}\left(-\frac{9}{10}\right)^n + \frac{33}{13}\left(\frac{1}{5}\right)^n.
\end{aligned}$$

ii) $0 \leq n \leq 1$,

$$\begin{aligned}
y(n) &= 2h(n) \\
&= 2\left[\frac{9}{13}\left(-\frac{9}{10}\right)^n + \frac{4}{13}\left(\frac{2}{5}\right)^n\right]u(n) \\
&= \frac{18}{13}\left(-\frac{9}{10}\right)^n + \frac{8}{13}\left(\frac{2}{5}\right)^n.
\end{aligned}$$

Therefore,

$$y(n) = \begin{cases} 0, \quad n < 0, \\ \frac{18}{13}\left(-\frac{9}{10}\right)^n + \frac{8}{13}\left(\frac{2}{5}\right)^n, \quad n = 0, 1, \\ \frac{262}{117}\left(-\frac{9}{10}\right)^n + \frac{33}{13}\left(\frac{2}{5}\right)^n, \quad n \geq 2. \end{cases}$$

(c) For what values of α is $g(n) = \alpha^n h(n)$ a finite energy sequence?

Solution:
The energy E_g of $g(n)$ is given by

$$\begin{aligned}
E_g &= \sum_{n=-\infty}^{\infty} |g(n)|^2 \\
&= \sum_{n=-\infty}^{\infty} |\alpha^n h(n)|^2 \\
&= \sum_{n=0}^{\infty} \left|\alpha^n\left\{\frac{9}{13}\left(-\frac{9}{10}\right)^n + \frac{4}{13}\left(\frac{2}{5}\right)^n\right\}\right|^2 \\
&= \sum_{n=0}^{\infty} \left|\frac{9}{13}\left(-\frac{9}{10}\alpha\right)^n + \frac{4}{13}\left(\frac{2}{5}\alpha\right)^n\right|^2 \\
&= \sum_{n=0}^{\infty} \frac{81}{169}\left(\frac{81}{100}\alpha^2\right)^n + \sum_{n=0}^{\infty} \frac{16}{169}\left(\frac{4}{25}\alpha^2\right)^n + \sum_{n=0}^{\infty} \frac{72}{169}\left(-\frac{9}{25}\alpha^2\right)^n.
\end{aligned}$$

To be a finite energy,

$$\left|\frac{81}{100}\alpha^2\right| < 1, \quad \left|\frac{4}{25}\alpha^2\right| < 1, \text{ and } \left|\frac{9}{25}\alpha^2\right| < 1,$$

or equivalently,

$$|\alpha| < \frac{10}{9}, \quad |\alpha| < \frac{5}{2}, \text{ and } |\alpha| < \frac{5}{3}.$$

Hence, α should have a value in the range of $|\alpha| < \frac{10}{9}$.

1. Consider the following two systems

$$y_1(n) = \frac{1}{10}\sum_{k=n-4}^{n} x(k),$$

$$y_2(n) = \begin{cases} x\left(\frac{n}{4}\right) & n = 4k, k \text{ integer} \\ 0 & n \neq 4k, k \text{ integer.} \end{cases}$$

(a) Determine whether the systems are linear, time-invariant, relaxed, BIBO stable, and causal. Justify your answer to receive full credit.

Solution:

Properties	$y_1(n)$	$y_2(n)$
Relaxed	Yes	No
Linear	Yes	Yes
Time-Invariant	Yes	No
BIBO Stable	Yes	Yes
Causal	Yes	No

- System $y_1(n)$:
  - $y_1(n) = \frac{1}{10}\sum_{k=n-4}^{n} x(k) = \frac{1}{10}[x(n-4) + x(n-3) + x(n-2) + x(n-1) + x(n)]$ can be considered as a relaxed constant coefficient difference equation. It is relaxed, linear, time-invariant, BIBO stable, and causal.
- System $y_2(n)$:
  - Not relaxed: Counter example: if the first non-zero sample of $x(n)$ is at $n = -1$, then $y_2(-4)$ will be non-zero, which is before -1.
  - Linear: Let two outputs be

$$y_2^a(n) = \begin{cases} x_a\left(\frac{n}{4}\right) & n = 4k, k \text{ integer} \\ 0 & n \neq 4k, k \text{ integer.} \end{cases},$$

and

$$y_2^b(n) = \begin{cases} x_b\left(\frac{n}{4}\right) & n = 4k, k \text{ integer} \\ 0 & n \neq 4k, k \text{ integer.} \end{cases},$$

for an input $x_a(n)$ and $x_b(n)$, respectively. The output $y(n)$ for an input $Ax_a(n)+Bx_b(n)$ with some constants A and B is given by
i) if $n = 4k$, k integer,

$$\begin{aligned} y(n) &= Ax_a\left(\frac{n}{3}\right) + Bx_b\left(\frac{n}{4}\right) \\ &= Ay_2^a(n) + By_2^b(n). \end{aligned}$$

ii) if $n \neq 4k$, k integer,

$$y(n) = 0.$$

Based on i) and ii), $y(n)$ for an input $Ax_a(n) + Bx_b(n)$ is

$$y(n) = Ay_2^a(n) + By_2^b(n),$$

which implies that system $y_2(n)$ is linear.

- BIBO Stable: For a bounded sequence $|x(n)| \leq M < \infty$ with a finite positive number M,

$$|y_2(n)| = \begin{cases} |x\left(\frac{n}{4}\right)| \leq M < \infty & n = 4k, k \text{ integer} \\ 0 < \infty & n \neq 4k, k \text{ integer.} \end{cases},$$

implying that $|y_2(n)|$ is bounded output.

- Not Time-Invariant: Consider inputs $\delta(n)$ and $\delta(n-1)$. The outputs are $\delta(n)$ and $\delta(n-4)$. This shows that $y_2(n)$ is not time-invariant.
- Not causal: For example, $y_2(-4)$ depends on $x(-1)$, which is on the future of $y_2(-4)$.

(b) Given $x(n) = nu(n)$, compute and plot $y_1(n)$ and $y_2(n)$ for $0 \leq n \leq 5$.

Solution:

n	0	1	2	3	4	5
$y_1(n)$	0	1/10	3/10	$6/10 = 3/5$	$10/10 = 1$	$15/10 = 3/2$
$y_2(n)$	0	0	0	0	1	0

The corresponding plots are shown in Figure 1.

(c) Find the z-transform of $y_1(n)$ and $y_2(n)$ in terms of the z-transform of $x(n)$.

Solution:
z-transform for $y_1(n) = \frac{1}{10}\sum_{k=n-4}^{n} x(k) = \frac{1}{10}[x(n-4) + x(n-3) + x(n-2) + x(n-1) + x(n)]$ is given by

$$\begin{aligned} Y_1(z) &= \frac{1}{10}[z^{-4}X(z) + z^{-3}X(z) + z^{-2}X(z) + z^{-1}X(z) + X(z)] \\ &= \frac{z^{-4} + z^{-3} + z^{-2} + z^{-1} + 1}{10}X(z). \end{aligned}$$

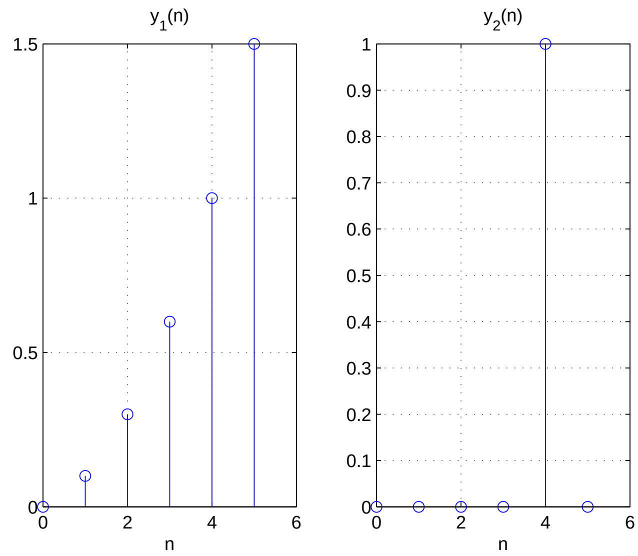
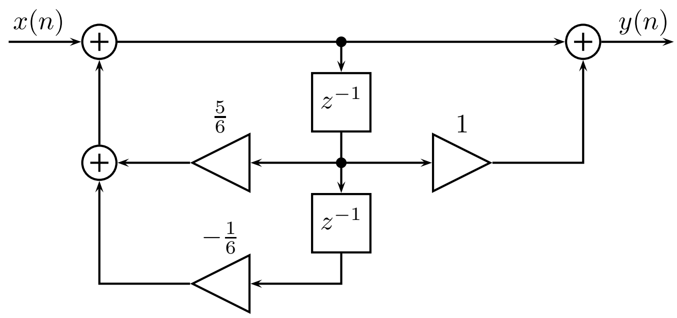


Figure 1: Plot for $y_1(n)$ and $y_2(n)$.

z-transform for $y_2(n)$ is given by

$$\begin{aligned} Y_2(z) &= \sum_{n=-\infty}^{\infty} y_2(n) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{4}\right) z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x(k) z^{-4k} \\ &= \sum_{k=-\infty}^{\infty} x(k) (z^4)^{-k} \\ &= X(z^4). \end{aligned}$$

2. A relaxed system is described by the following block diagram.



(a) Determine the constant coefficient difference equation that describes the system, and find its impulse response.

Solution:
Interchanging the order of the two systems in cascade, we get

$$y(n) = \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2) + x(n) + x(n-1).$$

Hence, the constant coefficient difference equation that describes the system is

$$y(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = x(n) + x(n-1).$$

The impulse response can be found by computing the zero-state response to $x(n) = \delta(n)$. We first find the homogeneous solution:

$$\lambda^2 - \frac{5}{6}\lambda + \frac{1}{6} = 0 \quad \Rightarrow \quad \lambda_1 = \frac{1}{2}, \quad \lambda_2 = \frac{1}{3}$$

and hence,

$$y_h(n) = C_1\left(\frac{1}{2}\right)^n + C_2\left(\frac{1}{3}\right)^n, \quad \text{for all } n.$$

Letting $x(n) = \delta(n)$ and assuming that the system is relaxed, we get $y(0) = x(0) = 1$ and $y(1) = \frac{5}{6}y(0) + x(1) + x(0) = \frac{11}{6}$. From the homogeneous solution, we get

$$\begin{aligned} y(0) &= C_1 + C_2 = 1 \\ y(1) &= \frac{1}{2}C_1 + \frac{1}{3}C_2 = \frac{11}{6} \end{aligned}$$

and solving for C_1 and C_2 yields $C_1 = 9$ and $C_2 = -8$, and consequently,

$$h(n) = \left[9\left(\frac{1}{2}\right)^n - 8\left(\frac{1}{3}\right)^n\right]u(n).$$

(b) Given $x(n) = 3^n u(-n)$, find the output of the system.

Solution:
Since the system is relaxed and described by a constant coefficient difference equation, it is LTI. Hence, the output is the convolution of the input and the impulse response as follows:

$$
\begin{aligned}
y(n) &= h(n) \star x(n) \\
&= \left\{\left[9\left(\frac{1}{2}\right)^n - 8\left(\frac{1}{3}\right)^n\right] u(n)\right\} \star [3^n u(-n)] \\
&= \left[9\left(\frac{1}{2}\right)^n u(n)\right] \star [3^n u(-n)] - \left[8\left(\frac{1}{3}\right)^n u(n)\right] \star [3^n u(-n)] \\
&= \sum_{k=-\infty}^{\infty} 9\left(\frac{1}{2}\right)^k u(k) 3^{n-k} u(-n+k) - \sum_{k=-\infty}^{\infty} 8\left(\frac{1}{3}\right)^k u(k) 3^{n-k} u(-n+k) \\
&= \sum_{k=\max\{0,n\}}^{\infty} 9\left(\frac{1}{2}\right)^k 3^{n-k} - \sum_{k=\max\{0,n\}}^{\infty} 8\left(\frac{1}{3}\right)^k 3^{n-k} \\
&= \begin{cases} \sum_{k=n}^{\infty} 9\left(\frac{1}{2}\right)^k 3^{n-k} - \sum_{k=n}^{\infty} 8\left(\frac{1}{3}\right)^k 3^{n-k} & n \geq 0 \\ \sum_{k=0}^{\infty} 9\left(\frac{1}{2}\right)^k 3^{n-k} - \sum_{k=0}^{\infty} 8\left(\frac{1}{3}\right)^k 3^{n-k} & n < 0 \end{cases} \\
&= \begin{cases} 9 \cdot 3^n \cdot \sum_{k=n}^{\infty} \left(\frac{1}{6}\right)^k - 8 \cdot 3^n \cdot \sum_{k=n}^{\infty} \left(\frac{1}{9}\right)^k & n \geq 0 \\ 9 \cdot 3^n \cdot \sum_{k=0}^{\infty} \left(\frac{1}{6}\right)^k - 8 \cdot 3^n \cdot \sum_{k=0}^{\infty} \left(\frac{1}{9}\right)^k & n < 0 \end{cases} \\
&= \begin{cases} 9 \cdot 3^n \cdot \left(\frac{1}{6}\right)^n \frac{1}{1-\frac{1}{6}} - 8 \cdot 3^n \cdot \left(\frac{1}{9}\right)^n \frac{1}{1-\frac{1}{9}} & n \geq 0 \\ 9 \cdot 3^n \cdot \frac{1}{1-\frac{1}{6}} - 8 \cdot 3^n \cdot \frac{1}{1-\frac{1}{9}} & n < 0 \end{cases} \\
&= \begin{cases} \frac{54}{5} \cdot \left(\frac{1}{2}\right)^n - 9 \cdot \left(\frac{1}{3}\right)^n & n \geq 0 \\ \frac{9}{5} \cdot 3^n & n < 0 \end{cases}
\end{aligned}
$$

3. The difference equation of a relaxed system is:

$$y(n) + 0.4y(n-1) - 0.21y(n-2) = x(n)$$

(a) Find a closed form for the impulse response $h(n)$ (i.e., the zero state system output when the input is an impulse).

Solution:
By setting $x(n) = \delta(n)$, we have

$$h(n) + 0.4h(n-1) - 0.21h(n-2) = \delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$

Hence, for $n \geq 1$, $h(n)$ can be found by solving homogeneous difference equation

$$h(n) + 0.4h(n-1) - 0.21h(n-2) = 0. \quad (1)$$

The characteristic polynomial for (1) can be solved by

$$\left(\lambda + \frac{7}{10}\right)\left(\lambda - \frac{3}{10}\right) = 0,$$

thus, the modes are

$$\lambda_1 = -\frac{7}{10}, \quad \lambda_2 = \frac{3}{10}.$$

Hence,

$$h(n) = C_1\left(-\frac{7}{10}\right)^n + C_2\left(\frac{3}{10}\right)^n, \quad n \geq 0.$$

Since the system is relaxed, $y(-1) = h(-1) = 0$ and $h(0) = \delta(0) = 1$, which gives

$$C_1 = \frac{7}{10}, \quad C_2 = \frac{3}{10}.$$

Therefore,

$$h(n) = \left[\frac{7}{10}\left(-\frac{7}{10}\right)^n + \frac{3}{10}\left(\frac{3}{10}\right)^n\right] u(n).$$

(b) If the input to the system is $x(n) = 3\delta(n) + \delta(n-2)$, what is the output?

Solution:
The output $y(n)$ of the system can be expressed as the convolution of $x(n)$ and $h(n)$, i.e.,

$$\begin{aligned} y(n) &= x(n) * h(n) \\ &= [3\delta(n) + \delta(n-2)] * h(n) \\ &= 3h(n) + h(n-2). \end{aligned}$$

Hence, using $h(n)$ in part (a) gives us
i) $n \geq 2$,

$$\begin{aligned}
y(n) &= 3h(n) + h(n-2) \\
&= 3\left[\frac{7}{10}\left(-\frac{7}{10}\right)^n + \frac{3}{10}\left(\frac{3}{10}\right)^n\right]u(n) + \left[\frac{7}{10}\left(-\frac{7}{10}\right)^{n-2} + \frac{3}{10}\left(\frac{3}{10}\right)^{n-2}\right]u(n-2) \\
&= \left[3\cdot\frac{7}{10} + \frac{7}{10}\cdot\left(\frac{7}{10}\right)^{-2}\right]\left(-\frac{7}{10}\right)^n + \left[3\cdot\frac{3}{10} + \frac{3}{10}\cdot\left(\frac{3}{10}\right)^{-2}\right]\left(\frac{3}{10}\right)^n \\
&= \frac{247}{70}\left(-\frac{7}{10}\right)^n + \frac{127}{30}\left(\frac{3}{10}\right)^n.
\end{aligned}$$

ii) $0 \leq n \leq 1$,

$$\begin{aligned}
y(n) &= 3h(n) \\
&= 3\left[\frac{7}{10}\left(-\frac{7}{10}\right)^n + \frac{3}{10}\left(\frac{3}{10}\right)^n\right]u(n) \\
&= \frac{21}{10}\left(-\frac{7}{10}\right)^n + \frac{9}{10}\left(\frac{3}{10}\right)^n.
\end{aligned}$$

Therefore,

$$y(n) = \begin{cases} 0, \quad n < 0, \\ \frac{21}{10}\left(-\frac{7}{10}\right)^n + \frac{9}{10}\left(\frac{3}{10}\right)^n, \quad n = 0, 1, \\ \frac{247}{70}\left(-\frac{7}{10}\right)^n + \frac{127}{30}\left(\frac{3}{10}\right)^n, \quad n \geq 2. \end{cases}$$

(c) For what values of α is $g(n) = \alpha^n h(n)$ a finite energy sequence?

Solution:
The energy E_g of $g(n)$ is given by

$$\begin{aligned}
E_g &= \sum_{n=-\infty}^{\infty} |g(n)|^2 \\
&= \sum_{n=-\infty}^{\infty} |\alpha^n h(n)|^2 \\
&= \sum_{n=0}^{\infty} \left|\alpha^n \left\{\frac{7}{10}\left(-\frac{7}{10}\right)^n + \frac{3}{10}\left(\frac{3}{10}\right)^n\right\}\right|^2 \\
&= \sum_{n=0}^{\infty} \left|\frac{7}{10}\left(-\frac{7}{10}\alpha\right)^n + \frac{3}{10}\left(\frac{3}{10}\alpha\right)^n\right|^2 \\
&= \sum_{n=0}^{\infty} \frac{49}{100}\left(\frac{49}{100}\alpha^2\right)^n + \sum_{n=0}^{\infty} \frac{21}{50}\left(-\frac{21}{100}\alpha^2\right)^n + \sum_{n=0}^{\infty} \frac{9}{100}\left(\frac{9}{100}\alpha^2\right)^n.
\end{aligned}$$

To be a finite energy,

$$\left|\frac{49}{100}\alpha^2\right| < 1, \quad \left|-\frac{21}{50}\alpha^2\right| < 1, \text{ and } \left|\frac{9}{100}\alpha^2\right| < 1,$$

or equivalently,

$$|\alpha| < \frac{10}{7}, \quad |\alpha| < \sqrt{\frac{50}{21}}, \text{ and } |\alpha| < \frac{10}{3}.$$

Hence, α should have a value in the range of $|\alpha| < \frac{10}{7}$.