EE113: Digital Signal Processing Spring 2011
Solution to Midterm Exam (Version A)

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1. Consider the following two systems

$$y_1(n) = \frac{1}{3}\sum_{k=n-2}^{n} x(k),$$

$$y_2(n) = \begin{cases} x\left(\frac{n}{4}\right) & n = 4k, k \text{ integer} \\ 0 & n \neq 4k, k \text{ integer.} \end{cases}$$

(a) Determine whether the systems are linear, time-invariant, relaxed, BIBO stable, and causal. Justify your answer to receive full credit.

Solution:

Properties	$y_1(n)$	$y_2(n)$
Relaxed	Yes	No
Linear	Yes	Yes
Time-Invariant	Yes	No
BIBO Stable	Yes	Yes
Causal	Yes	No

- System $y_1(n)$:
  - $y_1(n) = \frac{1}{3}\sum_{k=n-2}^{n} x(k) = \frac{1}{3}[x(n-2) + x(n-1) + x(n)]$ can be considered as a relaxed constant coefficient difference equation. It is relaxed, linear, time-invariant, BIBO stable, and causal.
- System $y_2(n)$:
  - Not relaxed: Counter example: if the first non-zero sample of $x(n)$ is at $n = -1$, then $y_2(-4)$ will be non-zero, which is before -1.
  - Linear: Let two outputs be

$$y_2^a(n) = \begin{cases} x_a\left(\frac{n}{4}\right) & n = 4k, k \text{ integer} \\ 0 & n \neq 4k, k \text{ integer.} \end{cases},$$

and

$$y_2^b(n) = \begin{cases} x_b\left(\frac{n}{4}\right) & n = 4k, k \text{ integer} \\ 0 & n \neq 4k, k \text{ integer.} \end{cases},$$

for an input $x_a(n)$ and $x_b(n)$, respectively. The output $y(n)$ for an input $Ax_a(n) + Bx_b(n)$ with some constants A and B is given by
i) if $n = 4k$, k integer,

$$\begin{aligned} y(n) &= Ax_a\left(\frac{n}{4}\right) + Bx_b\left(\frac{n}{4}\right) \\ &= Ay_2^a(n) + By_2^b(n). \end{aligned}$$

ii) if $n \neq 4k$, k integer,

$$y(n) = 0.$$

Based on i) and ii), $y(n)$ for an input $Ax_a(n) + Bx_b(n)$ is

$$y(n) = Ay_2^a(n) + By_2^b(n),$$

which implies that system $y_2(n)$ is linear.

· BIBO Stable: For a bounded sequence $|x(n)| \leq M < \infty$ with a finite positive number M,

$$|y_2(n)| = \begin{cases} |x\left(\frac{n}{4}\right)| \leq M < \infty & n = 4k, k \text{ integer} \\ 0 < \infty & n \neq 4k, k \text{ integer.} \end{cases},$$

implying that $|y_2(n)|$ is bounded output.

· Not Time-Invariant: Consider inputs $\delta(n)$ and $\delta(n-1)$. The outputs are $\delta(n)$ and $\delta(n-4)$. This shows that $y_2(n)$ is not time-invariant.

· Not causal: For example, $y_2(-4)$ depends on $x(-1)$, i.e., the output depends on the future value of the input.

(b) Given $x(n) = nu(n) + \delta(n^2)$, compute and plot $y_1(n)$ and $y_2(n)$ for $0 \leq n \leq 5$.

n	0	1	2	3	4	5
$y_1(n)$	1/3	2/3	4/3	6/3=2	9/3=3	12/3=4
$y_2(n)$	1	0	0	0	1	0

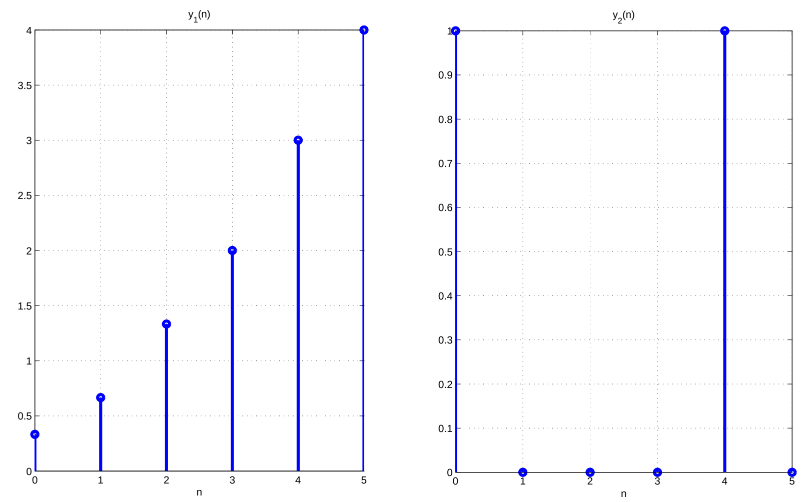

Figure 1: Plot for $y_1(n)$ and $y_2(n)$.

Solution:

The corresponding plots are shown in Figure 1.

(c) Find the z-transform of $y_1(n)$ and $y_2(n)$ in terms of the z-transform of $x(n)$.

Solution:
z-transform for $y_1(n) = \frac{1}{3}\sum_{k=n-2}^{n} x(k) = \frac{1}{3}[x(n-2) + x(n-1) + x(n)]$ gives

$$\begin{aligned} Y_1(z) &= \frac{1}{3}[z^{-2}X(z) + z^{-1}X(z) + X(z)] \\ &= \frac{z^{-2} + z^{-1} + 1}{3}X(z). \end{aligned}$$

z-transform for $y_2(n)$ is given by

$$\begin{aligned}
Y_2(z) &= \sum_{n=-\infty}^{\infty} y_2(n) z^{-n} \\
&= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{4}\right) z^{-n} \\
&= \sum_{k=-\infty}^{\infty} x(k) z^{-4k} \\
&= \sum_{k=-\infty}^{\infty} x(k) (z^4)^{-k} \\
&= X(z^4).
\end{aligned}$$

It should also be correct if the students just use the property of the z-transform.

2. The difference equation of a relaxed system is:

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n)$$

(a) Find a closed form for the impulse response $h(n)$ (i.e., the zero state system output when the input is an impulse).

Solution:
By setting $x(n) = \delta(n)$, we have

$$h(n) - \frac{3}{4}h(n-1) + \frac{1}{8}h(n-2) = \delta(n) = \left\{ \begin{array}{l} 1,\ n = 0 \\ 0,\ n \neq 0 \end{array} \right. .$$

Hence, for $n \geq 1$, $h(n)$ can be found by solving homogeneous difference equation

$$h(n) - \frac{3}{4}h(n-1) + \frac{1}{8}h(n-2) = 0. \tag{1}$$

The characteristic polynomial for (1) can be solved by

$$\left(\lambda - \frac{1}{2}\right)\left(\lambda - \frac{1}{4}\right) = 0,$$

thus, the modes are

$$\lambda_1 = \frac{1}{2}, \quad \lambda_2 = \frac{1}{4}.$$

Hence,

$$h(n) = C_1\left(\frac{1}{2}\right)^n + C_2\left(\frac{1}{4}\right)^n, \quad n \geq 0.$$

Since the system is relaxed, $y(-1) = h(-1) = 0$ and $h(0) = \delta(0) = 1$, which gives

$$C_1 = 2, \quad C_2 = -1.$$

Therefore,

$$h(n) = \left[2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right]u(n).$$

(b) If the input to the system is $x(n) = u(n-3)$, what is the output?

Solution:
The output $y(n)$ of the system can be expressed as the convolution of $x(n)$ and $h(n)$, i.e.,

$$\begin{aligned} y(n) &= x(n) * h(n) \\ &= u(n-3) * h(n). \end{aligned}$$

Hence, using $h(n)$ in part (a) gives us
i) $n \leq 2$, $y(n) = 0$
ii) $n \geq 3$,

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\ &= \sum_{k=-\infty}^{\infty} u(k-3)\left[2\left(\frac{1}{2}\right)^{n-k} - \left(\frac{1}{4}\right)^{n-k}\right] u(n-k) \\ &= \sum_{k=3}^{n} \left[2\left(\frac{1}{2}\right)^{n-k} - \left(\frac{1}{4}\right)^{n-k}\right] \\ &= \sum_{k=0}^{n-3} \left[2\left(\frac{1}{2}\right)^{k} - \left(\frac{1}{4}\right)^{k}\right] \\ &= 2.\frac{1-\left(\frac{1}{2}\right)^{n-2}}{1-\frac{1}{2}} - \frac{1-\left(\frac{1}{4}\right)^{n-2}}{1-\frac{1}{4}} \\ &= \frac{8}{3} - 4\left(\frac{1}{2}\right)^{n-2} + \frac{4}{3}\left(\frac{1}{4}\right)^{n-2} \end{aligned}$$

(c) For what values of α is $g(n) = \alpha^n h(n)$ a finite energy sequence?

Solution:

The energy E_g of $g(n)$ is given by

$$\begin{aligned}
E_g &= \sum_{n=-\infty}^{\infty} |g(n)|^2 \\
&= \sum_{n=-\infty}^{\infty} |\alpha^n h(n)|^2 \\
&= \sum_{n=0}^{\infty} \left| \alpha^n \left\{ 2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n \right\} \right|^2 \\
&= \sum_{n=0}^{\infty} \left| 2\left(\frac{1}{2}\alpha\right)^n - \left(\frac{1}{4}\alpha\right)^n \right|^2 \\
&= \sum_{n=0}^{\infty} 4\left(\frac{1}{4}\alpha^2\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{16}\alpha^2\right)^n - \sum_{n=0}^{\infty} 4\left(\frac{1}{8}\alpha^2\right)^n.
\end{aligned}$$

To be a energy sequence, the energy must be finite, and hence,

$$\left|\frac{1}{4}\alpha^2\right| < 1, \quad \left|\frac{1}{16}\alpha^2\right| < 1, \text{ and } \left|\frac{1}{8}\alpha^2\right| < 1,$$

or equivalently,

$$|\alpha| < 2, \quad |\alpha| < 4, \text{ and } |\alpha| < \sqrt{8}.$$

Hence, α should have a value in the range of $|\alpha| < 2$.

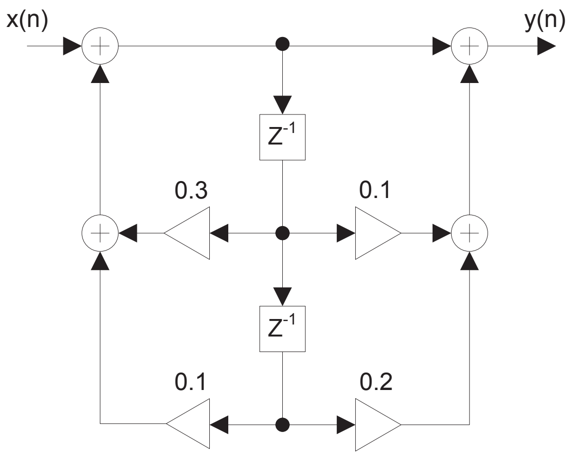

Figure 2: Block diagram.

3. A causal system is described by the above block diagram.

(a) Determine the constant coefficient difference equation that describes the system.

Solution:

We denote the intermediate signal as $w(n)$, shown in Fig. 3.

The system in Fig. 3 is equivalent to the system in the block diagram shown in Fig. 4.

The system in Fig. 4 is a cascade of two sections. Interchanging the order of them gives us the block diagram shown in Fig. 5.

In Fig. 5, the section on the left gives us

$$s(n) = x(n) + 0.1x(n-1) + 0.2x(n-2),$$

and the section on the right gives us

$$y(n) = s(n) + 0.3y(n-1) + 0.1y(n-2).$$

Combing the above two equations, we have

$$y(n) - 0.3y(n-1) - 0.1y(n-2) = x(n) + 0.1x(n-1) + 0.2x(n-2).$$

(b) Find the impulse response of the system.

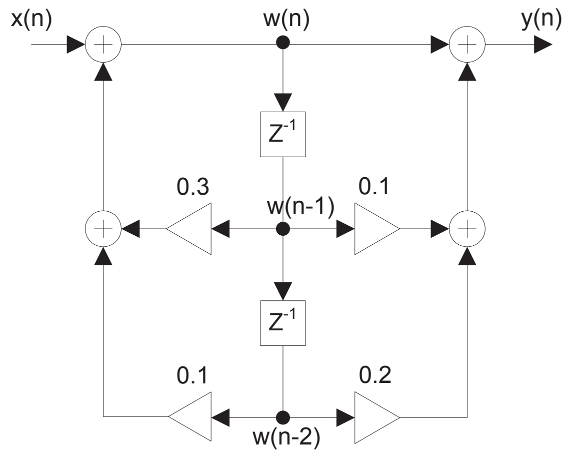

Figure 3: Block diagram with $w(n)$.

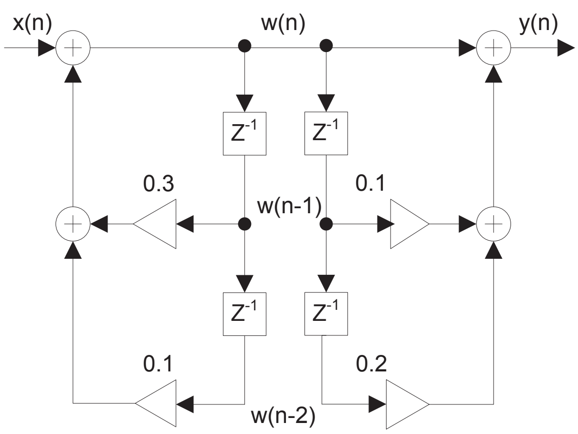

Figure 4: Equivalent block diagram.

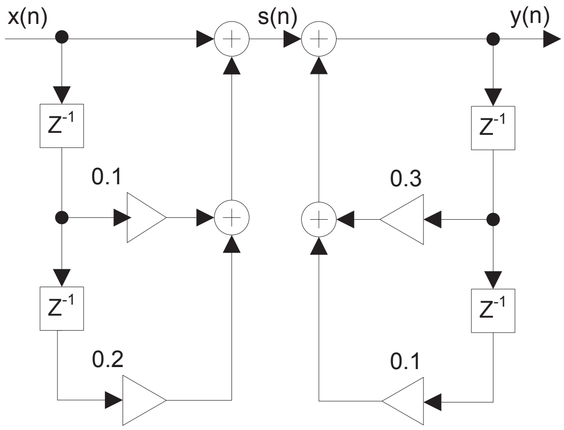

Figure 5: Equivalent block diagram with switched sections.

Solution:

The impulse response is the solution to the following equation:

$$h(n) - 0.3h(n-1) - 0.1h(n-2) = \delta(n) + 0.1\delta(n-1) + 0.2\delta(n-2).$$

When $n > 2$, the above difference equation is homogeneous, which has the following characteristic polynomial:

$$\lambda^2 - 0.3\lambda - 0.1 = 0.$$

Hence, the modes are

$$\lambda_1 = 0.5, \quad \lambda_2 = -0.2.$$

Thus, the solution is of the following form:

$$h(n) = C_1 \cdot (0.5)^n + C_2 \cdot (-0.2)^n.$$

Note that the equation is homogeneous when $n > 2$. Since the system is causal, we have $h(-2) = h(-1) = 0$, $h(0) = \delta(0) = 1$, $h(1) = 0.4$, and $h(2) = 0.42$. According to the two initial conditions

$$h(1) = C_1 \cdot 0.5 + C_2 \cdot (-0.2)$$

and

$$h(2) = C_1 \cdot 0.25 + C_2 \cdot 0.04,$$

we can solve for C_1 and C_2:

$$C_1 = \frac{10}{7}, \quad C_2 = \frac{11}{7}.$$

Finally, the impulse response is

$$h(n) = \delta(n) + \left[\frac{10}{7} \cdot (0.5)^n + \frac{11}{7} \cdot (-0.2)^n\right] u(n-1).$$

EE113: Digital Signal Processing Spring 2011
Solution to Midterm Exam (Version B)

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1. The difference equation of a relaxed system is:

$$y(n) - \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = x(n)$$

(a) Find a closed form for the impulse response $h(n)$ (i.e., the zero state system output when the input is an impulse).

<u>Solution</u>:
By setting $x(n) = \delta(n)$, we have

$$h(n) - \frac{1}{6}h(n-1) - \frac{1}{6}h(n-2) = \delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$

Hence, for $n \geq 1$, $h(n)$ can be found by solving homogeneous difference equation

$$h(n) - \frac{1}{6}h(n-1) - \frac{1}{6}h(n-2) = 0. \quad (1)$$

The characteristic polynomial for (1) can be solved by

$$\left(\lambda - \frac{1}{2}\right)\left(\lambda + \frac{1}{3}\right) = 0,$$

thus, the modes are

$$\lambda_1 = \frac{1}{2}, \quad \lambda_2 = -\frac{1}{3}.$$

Hence,

$$h(n) = C_1\left(\frac{1}{2}\right)^n + C_2\left(-\frac{1}{3}\right)^n, \quad n \geq 0.$$

Since the system is relaxed, $y(-1) = h(-1) = 0$ and $h(0) = \delta(0) = 1$, which gives

$$C_1 = \frac{3}{5}, \quad C_2 = \frac{2}{5}.$$

Therefore,

$$h(n) = \left[\frac{3}{5}\left(\frac{1}{2}\right)^n + \frac{2}{5}\left(-\frac{1}{3}\right)^n\right]u(n).$$

(b) If the input to the system is $x(n) = u(n-2)$, what is the output?

Solution:
The output $y(n)$ of the system can be expressed as the convolution of $x(n)$ and $h(n)$, i.e.,

$$\begin{aligned} y(n) &= x(n) * h(n) \\ &= u(n-2) * h(n). \end{aligned}$$

Hence, using $h(n)$ in part (a) gives us
i) $n \leq 1$, $y(n) = 0$
ii) $n \geq 2$,

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\ &= \sum_{k=-\infty}^{\infty} u(k-2)\left[\frac{3}{5}\left(\frac{1}{2}\right)^{n-k} + \frac{2}{5}\left(-\frac{1}{3}\right)^{n-k}\right]u(n-k) \\ &= \sum_{k=2}^{n}\left[\frac{3}{5}\left(\frac{1}{2}\right)^{n-k} + \frac{2}{5}\left(-\frac{1}{3}\right)^{n-k}\right] \\ &= \sum_{k=0}^{n-2}\left[\frac{3}{5}\left(\frac{1}{2}\right)^{k} + \frac{2}{5}\left(-\frac{1}{3}\right)^{k}\right] \\ &= \frac{3}{5}\cdot\frac{1-\left(\frac{1}{2}\right)^{n-1}}{1-\frac{1}{2}} + \frac{2}{5}\cdot\frac{1-\left(-\frac{1}{3}\right)^{n-1}}{1+\frac{1}{3}} \\ &= \frac{3}{2} - \frac{6}{5}\left(\frac{1}{2}\right)^{n-1} - \frac{3}{10}\left(-\frac{1}{3}\right)^{n-1} \end{aligned}$$

(c) For what values of α is $g(n) = \alpha^n h(n)$ a finite energy sequence?

<u>Solution</u>:
The energy E_g of $g(n)$ is given by

$$\begin{aligned}
E_g &= \sum_{n=-\infty}^{\infty} |g(n)|^2 \\
&= \sum_{n=-\infty}^{\infty} |\alpha^n h(n)|^2 \\
&= \sum_{n=0}^{\infty} \left| \alpha^n \left\{ \frac{3}{5}\left(\frac{1}{2}\right)^n + \frac{2}{5}\left(-\frac{1}{3}\right)^n \right\} \right|^2 \\
&= \sum_{n=0}^{\infty} \left| \frac{3}{5}\left(\frac{1}{2}\alpha\right)^n + \frac{2}{5}\left(-\frac{1}{3}\alpha\right)^n \right|^2 \\
&= \sum_{n=0}^{\infty} \frac{9}{25}\left(\frac{1}{4}\alpha^2\right)^n + \sum_{n=0}^{\infty} \frac{4}{25}\left(\frac{1}{9}\alpha^2\right)^n + \sum_{n=0}^{\infty} \frac{12}{25}\left(-\frac{1}{6}\alpha^2\right)^n .
\end{aligned}$$

To be a energy sequence, the energy must be finite, and hence,

$$\left|\frac{1}{4}\alpha^2\right| < 1, \quad \left|\frac{1}{9}\alpha^2\right| < 1, \text{ and } \left|-\frac{1}{6}\alpha^2\right| < 1,$$

or equivalently,

$$|\alpha| < 2, \quad |\alpha| < 3, \text{ and } |\alpha| < \sqrt{6}.$$

Hence, α should have a value in the range of $|\alpha| < 2$.

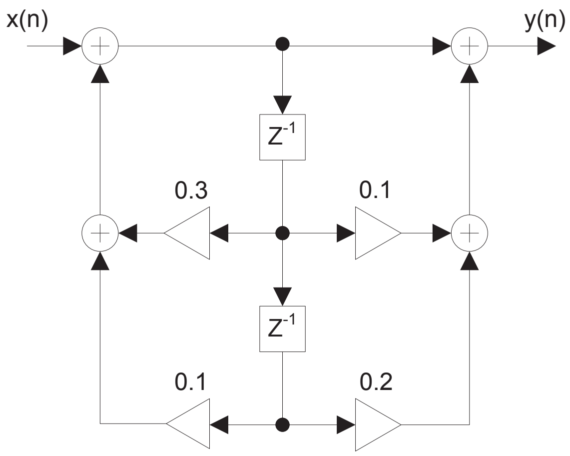

Figure 1: Block diagram.

2. A causal system is described by the above block diagram.

(a) Determine the constant coefficient difference equation that describes the system.

<u>Solution</u>:

We denote the intermediate signal as $w(n)$, shown in Fig. 2.

The system in Fig. 2 is equivalent to the system in the block diagram shown in Fig. 3.

The system in Fig. 3 is a cascade of two sections. Interchanging the order of them gives us the block diagram shown in Fig. 4.

In Fig. 4, the section on the left gives us

$$s(n) = x(n) + 0.1x(n-1) + 0.2x(n-2),$$

and the section on the right gives us

$$y(n) = s(n) + 0.3y(n-1) + 0.1y(n-2).$$

Combing the above two equations, we have

$$y(n) - 0.3y(n-1) - 0.1y(n-2) = x(n) + 0.1x(n-1) + 0.2x(n-2).$$

(b) Find the impulse response of the system.

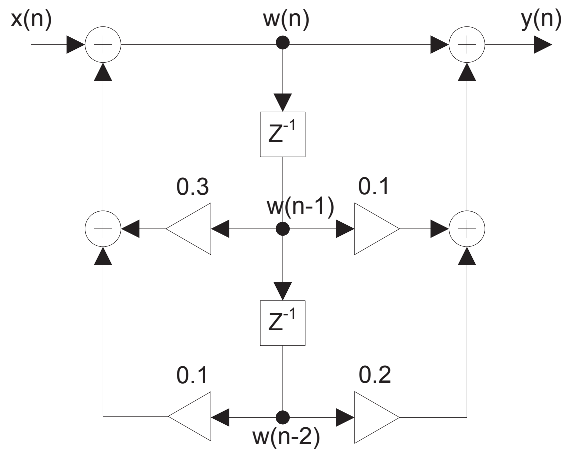

Figure 2: Block diagram with $w(n)$.

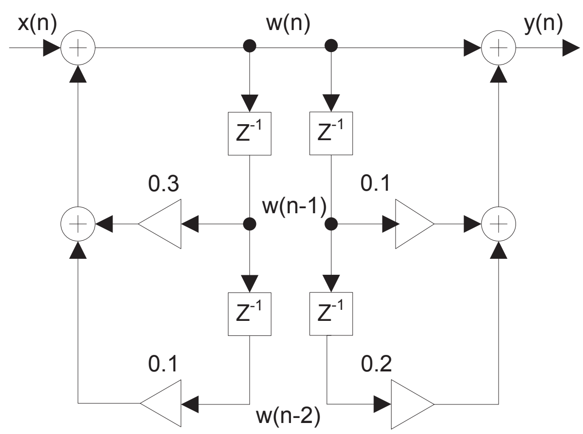

Figure 3: Equivalent block diagram.

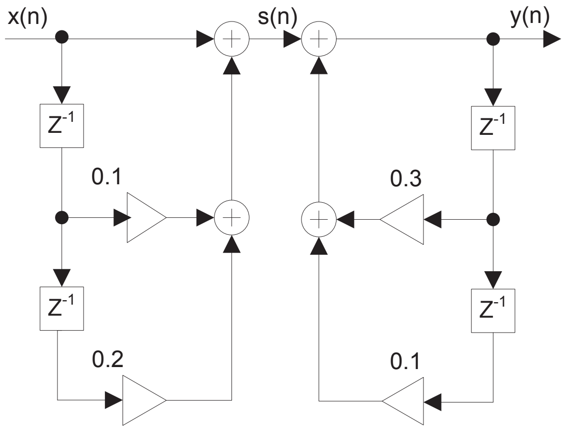

Figure 4: Equivalent block diagram with switched sections.

Solution:

The impulse response is the solution to the following equation:

$$h(n) - 0.3h(n-1) - 0.1h(n-2) = \delta(n) + 0.1\delta(n-1) + 0.2\delta(n-2).$$

When $n > 2$, the above difference equation is homogeneous, which has the following characteristic polynomial:

$$\lambda^2 - 0.3\lambda - 0.1 = 0.$$

Hence, the modes are

$$\lambda_1 = 0.5, \quad \lambda_2 = -0.2.$$

Thus, the solution is of the following form:

$$h(n) = C_1 \cdot (0.5)^n + C_2 \cdot (-0.2)^n.$$

Note that the equation is homogeneous when $n > 2$. Since the system is causal, we have $h(-2) = h(-1) = 0$, $h(0) = \delta(0) = 1$, $h(1) = 0.4$, and $h(2) = 0.42$. According to the two initial conditions

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$$h(2) = C_1 \cdot 0.25 + C_2 \cdot 0.04,$$

we can solve for C_1 and C_2:

$$C_1 = \frac{10}{7}, \quad C_2 = \frac{11}{7}.$$

Finally, the impulse response is

$$h(n) = \delta(n) + \left[\frac{10}{7} \cdot (0.5)^n + \frac{11}{7} \cdot (-0.2)^n\right] u(n-1).$$

3. Consider the following two systems

$$y_1(n) = \frac{1}{4} \sum_{k=n-3}^{n} x(k),$$

$$y_2(n) = \begin{cases} x\left(\frac{n}{2}\right) & n = 2k, k \text{ integer} \\ 0 & n \neq 2k, k \text{ integer.} \end{cases}$$

(a) Determine whether the systems are linear, time-invariant, relaxed, BIBO stable, and causal. Justify your answer to receive full credit.

Solution:

Properties	$y_1(n)$	$y_2(n)$
Relaxed	Yes	No
Linear	Yes	Yes
Time-Invariant	Yes	No
BIBO Stable	Yes	Yes
Causal	Yes	No

- System $y_1(n)$:
  - · $y_1(n) = \frac{1}{4}\sum_{k=n-3}^{n} x(k) = \frac{1}{4}[x(n-3)+x(n-2)+x(n-1)+x(n)]$ can be considered as a relaxed constant coefficient difference equation. It is relaxed, linear, time-invariant, BIBO stable, and causal.
- System $y_2(n)$:
  - · Not relaxed: Counter example: if the first non-zero sample of $x(n)$ is at $n = -1$, then $y_2(-2)$ will be non-zero, which is before -1.
  - · Linear: Let two outputs be

$$y_2^a(n) = \begin{cases} x_a\left(\frac{n}{2}\right) & n = 2k, k \text{ integer} \\ 0 & n \neq 2k, k \text{ integer.} \end{cases},$$

and

$$y_2^b(n) = \begin{cases} x_b\left(\frac{n}{2}\right) & n = 2k, k \text{ integer} \\ 0 & n \neq 2k, k \text{ integer.} \end{cases},$$

for an input $x_a(n)$ and $x_b(n)$, respectively. The output $y(n)$ for an input $Ax_a(n) + Bx_b(n)$ with some constants A and B is given by
i) if $n = 2k$, k integer,

$$\begin{aligned} y(n) &= Ax_a\left(\frac{n}{2}\right) + Bx_b\left(\frac{n}{2}\right) \\ &= Ay_2^a(n) + By_2^b(n). \end{aligned}$$

ii) if $n \neq 2k$, k integer,

$$y(n) = 0.$$

Based on i) and ii), $y(n)$ for an input $Ax_a(n)+Bx_b(n)$ is

$$y(n) = Ay_2^a(n) + By_2^b(n),$$

which implies that system $y_2(n)$ is linear.

- BIBO Stable: For a bounded sequence $|x(n)| \leq M < \infty$ with a finite positive number M,

$$|y_2(n)| = \begin{cases} |x\left(\frac{n}{2}\right)| \leq M < \infty & n = 2k, k \text{ integer} \\ 0 < \infty & n \neq 2k, k \text{ integer}. \end{cases},$$

implying that $|y_2(n)|$ is bounded output.

- Not Time-Invariant: Consider inputs $\delta(n)$ and $\delta(n-1)$. The outputs are $\delta(n)$ and $\delta(n-2)$. This shows that $y_2(n)$ is not time-invariant.
- Not causal: For example, $y_2(-2)$ depends on $x(-1)$, which is on the future of $y_2(-2)$.

(b) Given $x(n) = nu(n) + \delta(n^3)$, compute and plot $y_1(n)$ and $y_2(n)$ for $0 \leq n \leq 5$.

Solution:

The corresponding plots are shown in Figure 5.

n	0	1	2	3	4	5
$y_1(n)$	1/4	2/4=1/2	4/4=1	7/4	10/4 = 5/2	14/4 = 7/2
$y_2(n)$	1	0	1	0	2	0

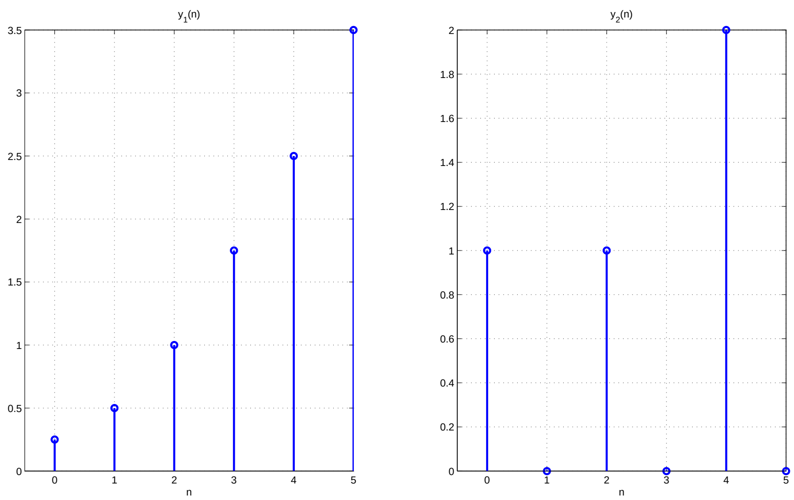

Figure 5: Plot for $y_1(n)$ and $y_2(n)$.

(c) Find the z-transform of $y_1(n)$ and $y_2(n)$ in terms of the z-transform of $x(n)$.

Solution:
z-transform for $y_1(n) = \frac{1}{4}\sum_{k=n-3}^{n} x(k) = \frac{1}{4}[x(n-3)+x(n-2)+x(n-1)+x(n)]$ gives

$$\begin{aligned} Y_1(z) &= \frac{1}{4}[z^{-3}X(z) + z^{-2}X(z) + z^{-1}X(z) + X(z)] \\ &= \frac{z^{-3}+z^{-2}+z^{-1}+1}{4}X(z). \end{aligned}$$

z-transform for $y_2(n)$ is given by

$$
\begin{aligned}
Y_2(z) &= \sum_{n=-\infty}^{\infty} y_2(n) z^{-n} \\
&= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2}\right) z^{-n} \\
&= \sum_{k=-\infty}^{\infty} x(k) z^{-2k} \\
&= \sum_{k=-\infty}^{\infty} x(k) (z^2)^{-k} \\
&= X(z^2).
\end{aligned}
$$

It should also be correct if the students just use the property of the z-transform.