EE113: Digital Signal Processing
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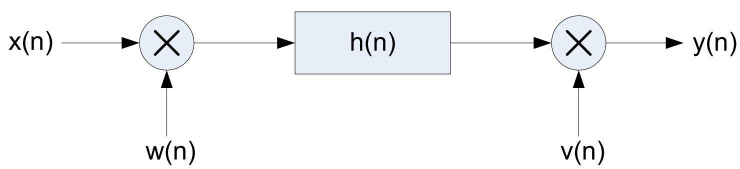
MIDTERM SOLUTIONS

1. **Problem 1**

A system is described by the block diagram shown in the figure below with $x(n)$ denoting the input sequence, $y(n)$ denoting the output sequence, $w(n) = e^{j\omega_0 n}$, and $v(n) = e^{-j\omega_0 n}$. $h(n)$ is known to be a linear, time-invariant, stable, and causal system.



(a) (10 PTS) Is the system linear?

The system is *linear.*

$$
\begin{aligned}
y(n) &= [x(n)w(n) * h(n)]\, v(n) \\
&= \left[x(n)e^{j\omega_0 n} * h(n)\right] e^{-j\omega_0 n} \\
&= \left[\sum_{k=-\infty}^{\infty} h(k)x(n-k)e^{j\omega_0 (n-k)}\right] e^{-j\omega_0 n} \\
&= \left[\sum_{k=-\infty}^{\infty} h(k)x(n-k)e^{j\omega_0 n}e^{-j\omega_0 k}\right] e^{-j\omega_0 n} \\
&= e^{j\omega_0 n}e^{-j\omega_0 n}\left[\sum_{k=-\infty}^{\infty} h(k)x(n-k)e^{-j\omega_0 k}\right] \\
&= \sum_{k=-\infty}^{\infty} h(k)x(n-k)e^{-j\omega_0 k}
\end{aligned}
$$

where the last line follows from the second-to-last because $e^{j\omega_0 n}e^{-j\omega_0 n} = 1$. We define $h_1(n) = h(n)e^{-j\omega_0 n}$. Since $y(n) = x(n) * h_1(n)$ the overall system S is basically a convolution, which we know is linear (see the derivation of the convolution sum in chapter 5 in the section titled 'The Convolution Sum'). To verify this, let $y_1(n) = x_1(n) * h_1(n) = S[x_1(n)]$ and $y_2(n) = x_2(n) * h_1(n) = S[x_2(n)]$. Then,

$$
\begin{aligned}
ay_1(n) + by_2(n) &= [ax_1(n) * h_1(n) + bx_2(n) * h_1(n)] \\
&= [ax_1(n) + bx_2(n)] * h_1(n) \\
&= S[ax_1(n) + bx_2(n)].
\end{aligned}
$$

Therefore, S is a linear operation.

(b) (10 PTS) Is the system time-invariant?

The system is *time-invariant*.
From part (a) we have

$$y(n) = S[x(n)] = \left[\sum_{k=-\infty}^{\infty} h(k)x(n-k)e^{-j\omega_0 k}\right].$$

Let

$$p(n) = S[x(n-n_0)] = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega_0 k}x(n-n_0-k).$$

Comparing $p(n)$ to

$$y(n-n_0) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega_0 k}x(n-n_0-k)$$

we see that $p(n) = y(n-n_0)$. Thus, S is time-invariant.

(c) (10 PTS) Is the system causal?

The system is *causal*.
Since S is LTI, S is causal if $h(n) = 0$ for $n < 0$. In this problem, as stated above, our impulse response is $h_1(n) = h(n)e^{-j\omega_0 n}$. S is causal if $h_1(n) = 0$ for $n < 0$. Since $h_1(n) = h(n)e^{-j\omega_0 n}$ and $h(n)$ is already an LTI causal system, we know $h(n) = 0$ for $n < 0$ so it follows that $h_1(n) = 0$ for $n < 0$. Thus, S is causal.

(d) (10 PTS) Is the system BIBO stable?

Now that we know the overall system is LTI, we know that S is BIBO stable if:

$$\sum_{n=-\infty}^{\infty} |h_1(n)| < \infty,$$

where $h_1(n) = h(n)e^{-j\omega_0 n}$ is our overall system impulse response from part (a). We have

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |h_1(n)| &< \infty \\
\sum_{n=-\infty}^{\infty} |h(n)e^{-j\omega_0 n}| &< \infty \\
\sum_{n=-\infty}^{\infty} |h(n)||e^{-j\omega_0 n}| &< \infty \\
\sum_{n=-\infty}^{\infty} |h(n)| &< \infty \\
,
\end{aligned}$$

where the last line follows from the second-to-last because $|e^{-j\omega_0 n}| = 1$. The above inequalities are true because the problem states that $h(n)$ is LTI, stable, and causal. Thus, S is stable.

2. **Problem 2**

The z-transform of a causal sequence $x(n)$ is

$$X(z) = \frac{z}{(z^2 - z + \frac{1}{2})(z + 3/4)}.$$

(a) (10 PTS) Specify the region of convergence for $X(z)$.

Because $x(n)$ is causal, the region of convergence must contain the unit-circle. Therefore, $|z| > 3/4$ is the region of convergence.

(b) (10 PTS) Determine $Y(z)$, where $y(n) = x(-n+3)$.

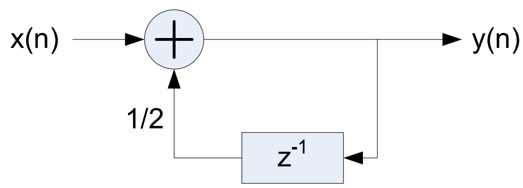
$$\begin{aligned} y(n) &= x(-n+3) = x(-(n-3)) \\ &\Rightarrow Y(z) = z^{-3}X(z^{-1}) = \frac{z^{-3}z^{-1}}{(z^{-2} - z^{-1} + \frac{1}{2})(z^{-1} + \frac{3}{4})} \\ &= \frac{8/3}{z(2 - 2z + z^2)(\frac{4}{3} + z)} \end{aligned}$$

(c) (10 PTS) Specify the region of convergence for $Y(z)$.

We know that $x(n)$ is a right-sided sequence because it is causal. Therefore, $x(-n+3)$ is left-sided, so the ROC must be $0 < |z| < 4/3$.

3. **Problem 3**

A causal system is described by the block diagram below with $x(n)$ denoting the input sequence and $y(n)$ denoting the output sequence.



(a) (10 PTS) Determine the modes of the system.

The system can be described by the equation $y(n) - \frac{1}{2}y(n-1) = x(n)$.
The characteristic equation is $\lambda - \frac{1}{2} = 0$. Therefore, the only mode of the system is $\lambda = \frac{1}{2}$.

(b) (10 PTS) Determine the impulse response of the system.

$y(n) = C(\frac{1}{2})^n$ is the solution to the homogeneous equation $y(n) - \frac{1}{2}y(n-1) = 0$.
Setting $x(n) = \delta(n)$ we get $h(n) - \frac{1}{2}h(n-1) = \delta(n)$. Therefore, $h(0) = \frac{1}{2}h(-1) + \delta(0) = 1$ ($h(-1) = 0$ because $h(n)$ is causal). This gives us $C = 1$ and $h(n) = (\frac{1}{2})^n u(n)$.

(c) (10 PTS) Determine the response of the system to the input

$$x(n) = \sum_{k=0}^{\infty} \delta(n-k) - 2u(n-n_1) + \sum_{k=n_2}^{\infty} \delta(n-k),$$

where $n_2 > n_1 > 0$, using convolution.

Note that $u(n) = \sum_{k=0}^{\infty} \delta(n-k)$ and $u(n-n_2) = \sum_{k=n_2}^{\infty} \delta(n-k)$. This means we can express $x(n)$ as

$$\begin{aligned} x(n) &= u(n) - 2u(n-n_1) + u(n-n_2) \\ &= [u(n) - u(n-n_1)] - [u(n-n_1) - u(n-n_2)] \end{aligned}$$

Thus,

$$\begin{aligned} y(n) &= [u(n) - 2u(n-n_1) + u(n-n_2)] * h(n) \\ &= u(n) * h(n) - 2u(n-n_1) * h(n) + u(n-n_2) * h(n). \end{aligned}$$

Let $y_1(n) = u(n) * h(n)$, $y_2(n) = -2u(n-n_1) * h(n)$, and $y_3(n) = u(n-n_2) * h(n)$. Each of these have the general form $y_i(n) = A_i u(n-n_i) * h(n)$, where A_i and n_i are integers. From discussion, we know that $y_i(n) = 0$ for $n < n_i$. Performing the convolution, we obtain:

$$\begin{aligned} y_i(n) &= \sum_{k=-\infty}^{\infty} h(k) A_i u(n - n_i - k) \\ &= A_i \sum_{k=0}^{\infty} h(k) u(n - n_i - k) \\ &= A_i \sum_{k=0}^{n-n_i} h(k) \\ &= A_i \sum_{k=0}^{n-n_i} (\frac{1}{2})^k \\ &= A_i \frac{1 - (\frac{1}{2})^{n-n_i+1}}{1 - \frac{1}{2}} \\ &= A_i \left[2 - 2(\frac{1}{2})^{n-n_i+1} \right] u(n - n_i) \end{aligned}$$

Finally,

$$\begin{aligned} y(n) &= y_1(n) + y_2(n) + y_3(n) \\ &= \left[2 - 2(\tfrac{1}{2})^{n+1}\right] u(n) - 2\left[2 - 2(\tfrac{1}{2})^{n-n_1+1}\right] u(n - n_1) + \left[2 - 2(\tfrac{1}{2})^{n-n_2+1}\right] u(n - n_2) \\ &= \begin{cases} 0, & n < 0 \\ 2 - (\frac{1}{2})^n, & 0 \le n \le n_1 - 1 \\ (\frac{1}{2})^{n-n_1-1} - (\frac{1}{2})^n - 2, & n_1 \le n \le n_2 - 1 \\ (\frac{1}{2})^{n-n_1-1} - (\frac{1}{2})^n - (\frac{1}{2})^{n-n_2}, & n \ge n_2. \end{cases} \end{aligned}$$