EE113: Digital Signal Processing Prof. Mihaela van der Schaar
Midterm Exam (Version A) Spring 2009

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1. A causal linear time-invariant system is initially relaxed and described by the difference equation
$$y(n) - 5y(n-1) + 6y(n-2) = 2x(n-1)$$
   (a) Determine the modes of the system.
   (b) Determine the impulse response of the system.
   (c) Determine the step response of the system using convolution.
   (d) Determine the step response of the system without using convolution.

2. (a) Consider the following complex series expansion of the natural logarithm for $|t| < 1$,
$$\ln(1+t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} t^n, \qquad |t| < 1$$
   Use the result to determine the sequence $x(n)$ whose z-transform is given by
$$X(z) = \ln(1 + \alpha z^{-1}), \qquad |z| > |\alpha|$$
   (b) Find the inverse z-transform of
$$X(z) = \frac{z^{-1}}{1 - (\frac{1}{2})^{50} z^{-50}}, \qquad |z| > \frac{1}{2}$$

3. Consider the system illustrated in Figure 1. The output of an LTI system with an impulse response $h(n) = (\frac{1}{4})^n u(n+10)$ is multiplied by a unit step function $u(n)$ to yield the output of the overall system. Answer each of the following questions, and briefly justify your answers:

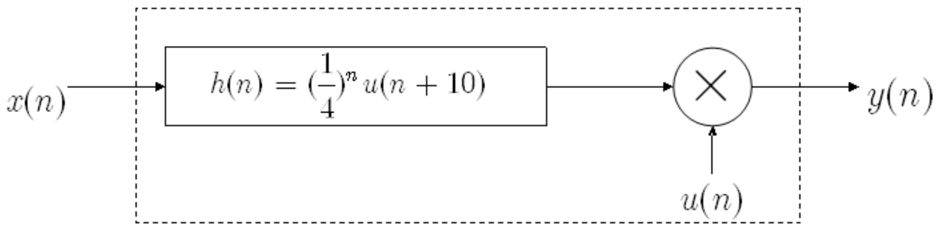

Figure 1: The overall system

   (a) Is the overall system linear?
   (b) Is the overall system time-invariant?
   (c) Is the overall system causal?
   (d) Is the overall system BIBO stable?

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1. (a) The modes are the roots of the equatoin $\lambda^2 - 5\lambda + 6 = 0$, which are $\lambda_1 = 2$ and $\lambda_2 = 3$.

   (b) The general homogeneous solution is $C_1 2^n + C_2 3^n$. The initial conditions are $h(0) = 0$ and $h(1) = 2$. We have

$$\begin{aligned} C_1 + C_2 &= 0 \\ 2C_1 + 3C_2 &= 2 \end{aligned}$$

   and the solutions are $C_1 = -2$ and $C_2 = 2$. The impulse response is $h(n) = [-2(2)^n + 2(3)^n]u(n)$.

   (c) The step response can be computed by

$$\begin{aligned} u(n) * h(n) &= \sum_{k=-\infty}^{\infty} u(n-k)[-2(2)^k + 2(3)^k]u(k) \\ &= \begin{cases} \sum_{k=0}^{n} [-2(2)^k + 2(3)^k], & n \geq 0 \\ 0, & n < 0 \end{cases} \\ &= \begin{cases} 3^{n+1} - 2(2)^{n+1} + 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \end{aligned}$$

   We can also write it as $[3^{n+1} - 2(2)^{n+1} + 1]u(n)$.

   (d) We can get the step response without using convolution. Let the step response be $w(n)$, then we first solve the particular solution, which is $w_p(n) = Ku(n)$. Substituting $w_p(n) = Ku(n)$ into the difference equation, we get

$$Ku(n) - 5Ku(n-1) + 6Ku(n-2) = 2u(n-1)$$

   For $n \geq 2$, we have $K = 1$ and hence $w_p(n) = u(n)$. The homogeneous solution is $w_h(n) = A_1(2)^n + A_2(3)^n$, hence the complete step response is $w(n) = w_p(n) + w_h(n)) = A_1(2)^n + A_2(3)^n + u(n)$. By using the initial conditions $w(0) = 0$ and $w(1) = 2$, we get

$$\begin{aligned} A_1 + A_2 + 1 &= 0 \\ 2A_1 + 3A_2 + 1 &= 2 \end{aligned}$$

   and the solutions are $A_1 = -4$ and $A_2 = 3$. We then have the step response as $w(n) = [-4(2)^n + 3(3)^n + u(n)]u(n)$ which can be also written as $w(n) = [3^{n+1} - 2(2)^{n+1} + 1]u(n)$. It is the same as the result we got by using convolution.

2. (a) We use the series expansion to get

$$\begin{aligned} X(z) &= \ln(1 + \alpha z^{-1}) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \alpha^n z^{-n} \end{aligned}$$

By comparing it to the definition of z-transform, which is

$$X(z) = \sum_{n=1}^{\infty} x(n) z^{-n}$$

we get $x(n) = \frac{(-1)^{n+1}}{n}\alpha^n$ for $n \geq 1$, and $x(n) = 0$ for $n < 1$. Hence, $x(n) = \frac{(-1)^{n+1}}{n}\alpha^n u(n-1)$.

(b) We can write $X(z)$ as

$$\begin{aligned} X(z) &= \frac{z^{-1}}{1-(\frac{1}{2})^{50} z^{-50}} \\ &= z^{-1}\sum_{k=0}^{\infty}(\frac{1}{2})^{50k} z^{-50k} \\ &= \sum_{k=0}^{\infty}(\frac{1}{2})^{50k} z^{-(50k+1)} \end{aligned}$$

By comparing it with the definition of z-transform, we get

$$x(n) = \begin{cases} (\frac{1}{2})^{n-1}, & n = 50k+1, k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

3. We let the output of the first sub-block (the LTI system) be $w(n)$. Hence, $y(n) = w(n)u(n)$.

(a) The system is linear.
Suppose $y_1(n)$ and $y_2(n)$ are the output sequences of the overall system to arbitrary input sequences $x_1(n)$ and $x_2(n)$, respectively. We also denote by $w_1(n)$ and $w_2(n)$ the outputs of the LTI system with impulse response $h(n)$. Hence, $y_1(n) = w_1(n)u(n)$ and $y_2(n) = w_2(n)u(n)$. Suppose the overall output to the input $x(n) = ax_1(n) + bx_2(n)$ is $y(n)$. Since the first sub-block is LTI, we know $w(n) = aw_1(n) + bw_2(n)$. Then we have

$$\begin{aligned} y(n) &= w(n)u(n) \\ &= (aw_1(n) + bw_2(n))u(n) \\ &= aw_1(n)u(n) + bw_2(n)u(n) \\ &= ay_1(n) + by_2(n) \end{aligned}$$

which proves the overall system is linear.

(b) The system is not time-invariant.
It can be shown by comparing the output sequences to input $x_1(n) = \delta(n)$ and $x_2(n) = x_1(n-4) = \delta(n-4)$. We have $y_1(n) = h(n)u(n) = (\frac{1}{4})^n u(n+10)u(n) = (\frac{1}{4})^n u(n)$ and $y_2(n) = h(n-4)u(n) = (\frac{1}{4})^{n-4} u(n+6)u(n) = (\frac{1}{4})^{n-4} u(n)$. Since $y_2(n) \neq y_1(n-4)$, we conclude that the overall system is not time-invariant.

(c) The system is not causal.

We note that the LTI system with impulse response $h(n) = (\frac{1}{4})^n u(n+10)$ is not causal. Therefore, any $y(n)$ for $n \geq 0$ depends on the input after n, which contradicts the definition of a causal system.

(d) The system is BIBO stable.

First we note that the LTI system with impulse response $h(n) = (\frac{1}{4})^n u(n+10)$ is BIBO stable because $\frac{1}{4} < 1$. Hence, for any bounded input $x(n)$ with $|x(n)| < \infty$ we have $|w(n)| < \infty$, and also $|y(n)| = |w(n)u(n)| < \infty$, which proves it is a BIBO stable system.