EE113: Digital Signal Processing Spring 2008
Solution to Midterm Exam (A)

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1. Consider the following two systems

$$y_1(n) = \frac{1}{4}\sum_{k=n-3}^{n} x(k),$$
$$y_2(n) = \begin{cases} x\left(\frac{n}{3}\right) & n = 3k, k \text{ integer} \\ 0 & n \neq 3k, k \text{ integer.} \end{cases}$$

(a) Determine whether the systems are linear, time-invariant, relaxed, BIBO stable, and causal. Justify your answer to receive full credit.

Solution:

Properties	$y_1(n)$	$y_2(n)$
Relaxed	Yes	No
Linear	Yes	Yes
Time-Invariant	Yes	No
BIBO Stable	Yes	Yes
Causal	Yes	No

- System $y_1(n)$:
  - $y_1(n) = \frac{1}{4}\sum_{k=n-3}^{n} x(k) = \frac{1}{4}[x(n-3) + x(n-2) + x(n-1) + x(n)]$ can be considered as a relaxed constant coefficient difference equation. It is relaxed, linear, time-invariant, BIBO stable, and causal.
- System $y_2(n)$:
  - Not relaxed: Counter example: if the first non-zero sample of $x(n)$ is at $n = -1$, then $y_2(-3)$ will be non-zero, which is before -1.
  - Linear: Let two outputs be

$$y_2^a(n) = \begin{cases} x_a\left(\frac{n}{3}\right) & n = 3k, k \text{ integer} \\ 0 & n \neq 3k, k \text{ integer.} \end{cases},$$

and

$$y_2^b(n) = \begin{cases} x_b\left(\frac{n}{3}\right) & n = 3k, k \text{ integer} \\ 0 & n \neq 3k, k \text{ integer.} \end{cases},$$

for an input $x_a(n)$ and $x_b(n)$, respectively. The output $y(n)$ for an input $Ax_a(n)+Bx_b(n)$ with some constants A and B is given by
i) if $n = 3k$, k integer,

$$y(n) = Ax_a\left(\frac{n}{3}\right) + Bx_b\left(\frac{n}{3}\right)$$
$$= Ay_2^a(n) + By_2^b(n).$$

ii) if $n \neq 3k$, k integer,

$$y(n) = 0.$$

Based on i) and ii), $y(n)$ for an input $Ax_a(n) + Bx_b(n)$ is

$$y(n) = Ay_2^a(n) + By_2^b(n),$$

which implies that system $y_2(n)$ is linear.

- BIBO Stable: For a bounded sequence $|x(n)| \leq M < \infty$ with a finite positive number M,

$$|y_2(n)| = \begin{cases} |x\left(\frac{n}{3}\right)| \leq M < \infty & n = 3k, k \text{ integer} \\ 0 < \infty & n \neq 3k, k \text{ integer}. \end{cases},$$

implying that $|y_2(n)|$ is bounded output.

- Not Time-Invariant: Consider inputs $\delta(n)$ and $\delta(n-1)$. The outputs are $\delta(n)$ and $\delta(n-3)$. This shows that $y_2(n)$ is not time-invariant.
- Not causal: For example, $y_2(-3)$ depends on $x(-1)$, which is on the future of $y_2(-3)$.

(b) Given $x(n) = nu(n)$, compute and plot $y_1(n)$ and $y_2(n)$ for $0 \leq n \leq 5$.

Solution:

n	0	1	2	3	4	5
$y_1(n)$	0	1/4	3/4	$6/4 = 3/2$	$10/4 = 5/2$	$14/4 = 7/2$
$y_2(n)$	0	0	0	1	0	0

The corresponding plots are shown in Figure 1.

(c) Find the z-transform of $y_1(n)$ and $y_2(n)$ in terms of the z-transform of $x(n)$.

Solution:

z-transform for $y_1(n) = \frac{1}{4}\sum_{k=n-3}^{n} x(k) = \frac{1}{4}[x(n-3) + x(n-2) + x(n-1) + x(n)]$ gives

$$\begin{aligned} Y_1(z) &= \frac{1}{4}[z^{-3}X(z) + z^{-2}X(z) + z^{-1}X(z) + X(z)] \\ &= \frac{z^{-3} + z^{-2} + z^{-1} + 1}{4}X(z). \end{aligned}$$

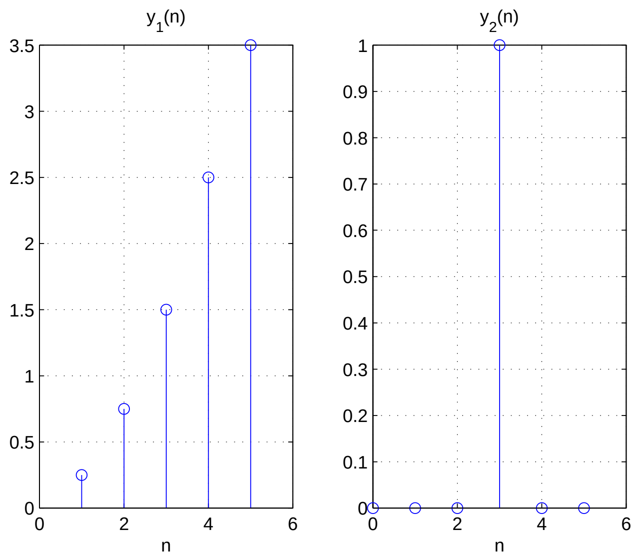
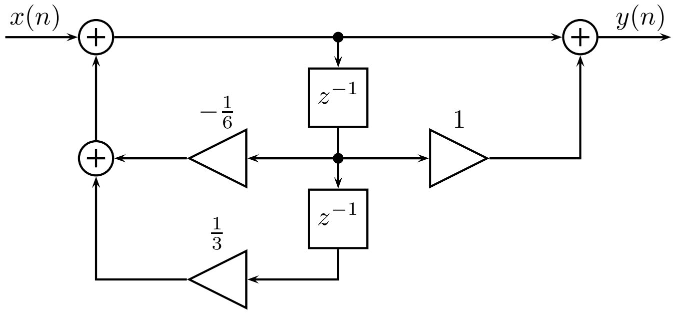


Figure 1: Plot for $y_1(n)$ and $y_2(n)$.

z-transform for $y_2(n)$ is given by

$$
\begin{aligned}
Y_2(z) &= \sum_{n=-\infty}^{\infty} y_2(n) z^{-n} \\
&= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{3}\right) z^{-n} \\
&= \sum_{k=-\infty}^{\infty} x(k) z^{-3k} \\
&= \sum_{k=-\infty}^{\infty} x(k) (z^3)^{-k} \\
&= X(z^3).
\end{aligned}
$$

2. A causal system is described by the following block diagram.



(a) Determine the constant coefficient difference equation that describes the system, and find its impulse response.

Solution:
Interchanging the order of the two systems in cascade, we get

$$y(n) = -\frac{1}{6}y(n-1) + \frac{1}{3}y(n-2) + x(n) + x(n-1).$$

The impulse response can be found by computing the zero-state response to $x(n) = \delta(n)$. We first find the homogeneous solution:

$$\lambda^2 + \frac{1}{6}\lambda - \frac{1}{3} = 0 \quad \Rightarrow \quad \lambda_1 = \frac{1}{2}, \quad \lambda_2 = -\frac{2}{3}$$

and hence,

$$y_h(n) = C_1\left(\frac{1}{2}\right)^n + C_2\left(-\frac{2}{3}\right)^n, \quad \text{for all } n.$$

Letting $x(n) = \delta(n)$ and assuming that the system is relaxed, we get $y(0) = 1$ and $y(1) = -\frac{1}{6}y(0) + 1 = \frac{5}{6}$. From the homogeneous solution we get

$$y(0) = C_1 + C_2 = 1$$
$$y(1) = \frac{1}{2}C_1 - \frac{2}{3}C_2 = \frac{5}{6}$$

and solving for C_1 and C_2 yields $C_1 = \frac{9}{7}$ and $C_2 = -\frac{2}{7}$, and consequently,

$$h(n) = \left[\frac{9}{7}\left(\frac{1}{2}\right)^n - \frac{2}{7}\left(-\frac{2}{3}\right)^n\right]u(n).$$

(b) Given $x(n) = n2^n u(-n)$, find the output of the system using the z-transform.

Solution:
The output $y(n)$ when $x(n) = n2^n u(-n)$ can be found by computing the inverse z-transform of $Y(z) = H(z)X(z)$. The z-transform of $h(n)$ is readily found from the impulse response:

$$H(z) = \frac{1 + z^{-1}}{1 + \frac{1}{6}z^{-1} - \frac{1}{3}z^{-2}}, \quad \text{ROC: } |z| > \frac{2}{3}.$$

The z-transform of $x(n)$ is given by

$$X(z) = \mathcal{Z}\{x(n)\} = -z\frac{d}{dz}\mathcal{Z}\{2^n u(-n)\} = -z\frac{d}{dz}\left[\sum_{n=-\infty}^{0} 2^n z^{-n}\right]$$

$$= -z\frac{d}{dz}\left[\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right] = -z\frac{d}{dz}\left[\frac{1}{1-\frac{1}{2}z}\right] = -\frac{2z}{(2-z)^2}, \quad \text{ROC: } |z| < 2.$$

Consequently, $Y(z)$ is given by

$$Y(z) = H(z)X(z) = \frac{-2z^3 - 2z^2}{(z-\frac{1}{2})(z+\frac{2}{3})(z-2)^2}, \quad \text{ROC: } \frac{2}{3} < |z| < 2.$$

Using partial fraction expansion, we get

$$\frac{Y(z)}{z} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z+\frac{2}{3}} + \frac{C}{z-2} + \frac{D}{(z-2)^2}$$

where

$$A = \left(z-\frac{1}{2}\right)\frac{Y(z)}{z}\bigg|_{z=\frac{1}{2}} = -\frac{4}{7}$$

$$B = \left(z+\frac{2}{3}\right)\frac{Y(z)}{z}\bigg|_{z=-\frac{2}{3}} = -\frac{3}{56}$$

$$C = \frac{d}{dz}\left[(z-2)^2\frac{Y(z)}{z}\right]\bigg|_{z=2} = \frac{5}{8}$$

$$D = (z-2)^2\frac{Y(z)}{z}\bigg|_{z=2} = -3.$$

Thus,

$$Y(z) = -\frac{4}{7}\frac{z}{z-\frac{1}{2}} - \frac{3}{56}\frac{z}{z+\frac{2}{3}} + \frac{5}{8}\frac{z}{z-2} - \frac{3}{2}\frac{2z}{(z-2)^2}$$

and since the ROC of $Y(z)$ is given by $\frac{2}{3} < |z| < 2$, we get

$$\begin{aligned} \frac{z}{z-\frac{1}{2}} &\leftrightarrow \left(\frac{1}{2}\right)^n u(n) \\ \frac{z}{z+\frac{2}{3}} &\leftrightarrow \left(-\frac{2}{3}\right)^n u(n) \\ \frac{z}{z-2} &\leftrightarrow -2^n u(-n-1) \\ \frac{2z}{(z-2)^2} &\leftrightarrow -n2^n u(-n-1) \end{aligned}$$

and consequently,

$$y(n) = \left[-\frac{4}{7}\left(\frac{1}{2}\right)^n - \frac{3}{56}\left(-\frac{2}{3}\right)^n\right]u(n) + \left[-\frac{5}{8} + \frac{3}{2}n\right]2^n u(-n-1).$$

Alternatively, partial fraction expansion on $Y(z)$ yields

$$y(n) = \left[-\frac{2}{7}\left(\frac{1}{2}\right)^{n-1} + \frac{1}{28}\left(-\frac{2}{3}\right)^{n-1}\right]u(n-1) + \left[-\frac{5}{4} + 3n\right]2^{n-1}u(-n).$$

3. The difference equation of a relaxed system is:

$$y(n) + 0.5y(n-1) - 0.14y(n-2) = x(n)$$

(a) Find a closed form for the impulse response $h(n)$ (i.e., the zero state system output when the input is an impulse).

<u>Solution</u>:
By setting $x(n) = \delta(n)$, we have

$$h(n) + 0.5h(n-1) - 0.14h(n-2) = \delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases}$$

Hence, for $n \geq 1$, $h(n)$ can be found by solving homogeneous difference equation

$$h(n) + 0.5h(n-1) - 0.14h(n-2) = 0. \tag{1}$$

The characteristic polynomial for (1) can be solved by

$$\left(\lambda + \frac{7}{10}\right)\left(\lambda - \frac{1}{5}\right) = 0,$$

thus, the modes are

$$\lambda_1 = -\frac{7}{10}, \quad \lambda_2 = \frac{1}{5}.$$

Hence,

$$h(n) = C_1\left(-\frac{7}{10}\right)^n + C_2\left(\frac{1}{5}\right)^n, \quad n \geq 0.$$

Since the system is relaxed, $y(-1) = h(-1) = 0$ and $h(0) = \delta(0) = 1$, which gives

$$C_1 = \frac{7}{9}, \quad C_2 = \frac{2}{9}.$$

Therefore,

$$h(n) = \left[\frac{7}{9}\left(-\frac{7}{10}\right)^n + \frac{2}{9}\left(\frac{1}{5}\right)^n\right]u(n).$$

(b) If the input to the system is $x(n) = 2\delta(n) + \delta(n-2)$, what is the output?

<u>Solution</u>:
The output $y(n)$ of the system can be expressed as the convolution of $x(n)$ and $h(n)$, i.e.,

$$\begin{aligned} y(n) &= x(n) * h(n) \\ &= [2\delta(n) + \delta(n-2)] * h(n) \\ &= 2h(n) + h(n-2). \end{aligned}$$

Hence, using $h(n)$ in part (a) gives us
i) $n \geq 2$,

$$\begin{aligned}
y(n) &= 2h(n) + h(n-2) \\
&= 2\left[\frac{7}{9}\left(-\frac{7}{10}\right)^n + \frac{2}{9}\left(\frac{1}{5}\right)^n\right]u(n) + \left[\frac{7}{9}\left(-\frac{7}{10}\right)^{n-2} + \frac{2}{9}\left(\frac{1}{5}\right)^{n-2}\right]u(n-2) \\
&= \left[2\cdot\frac{7}{9} + \frac{7}{9}\cdot\left(\frac{7}{10}\right)^{-2}\right]\left(-\frac{7}{10}\right)^n + \left[2\cdot\frac{2}{9} + \frac{2}{9}\cdot\left(\frac{1}{5}\right)^{-2}\right]\left(\frac{1}{5}\right)^n \\
&= \frac{22}{7}\left(-\frac{7}{10}\right)^n + 6\left(\frac{1}{5}\right)^n.
\end{aligned}$$

ii) $0 \leq n \leq 1$,

$$\begin{aligned}
y(n) &= 2h(n) \\
&= 2\left[\frac{7}{9}\left(-\frac{7}{10}\right)^n + \frac{2}{9}\left(\frac{1}{5}\right)^n\right]u(n) \\
&= \frac{14}{9}\left(-\frac{7}{10}\right)^n + \frac{4}{9}\left(\frac{1}{5}\right)^n.
\end{aligned}$$

Therefore,

$$y(n) = \begin{cases} 0, \quad n < 0, \\ \frac{14}{9}\left(-\frac{7}{10}\right)^n + \frac{4}{9}\left(\frac{1}{5}\right)^n, & n = 0, 1, \\ \frac{22}{7}\left(-\frac{7}{10}\right)^n + 6\left(\frac{1}{5}\right)^n, & n \geq 2. \end{cases}$$

(c) For what values of α is $g(n) = \alpha^n h(n)$ a finite energy sequence?

Solution:
The energy E_g of $g(n)$ is given by

$$\begin{aligned}
E_g &= \sum_{n=-\infty}^{\infty} |g(n)|^2 \\
&= \sum_{n=-\infty}^{\infty} |\alpha^n h(n)|^2 \\
&= \sum_{n=0}^{\infty}\left|\alpha^n\left\{\frac{7}{9}\left(-\frac{7}{10}\right)^n + \frac{2}{9}\left(\frac{1}{5}\right)^n\right\}\right|^2 \\
&= \sum_{n=0}^{\infty}\left|\frac{7}{9}\left(-\frac{7}{10}\alpha\right)^n + \frac{2}{9}\left(\frac{1}{5}\alpha\right)^n\right|^2 \\
&= \sum_{n=0}^{\infty}\frac{49}{81}\left(\frac{49}{100}\alpha^2\right)^n + \sum_{n=0}^{\infty}\frac{4}{81}\left(\frac{1}{25}\alpha^2\right)^n + \sum_{n=0}^{\infty}\frac{28}{81}\left(-\frac{7}{50}\alpha^2\right)^n.
\end{aligned}$$

Thus, for the energy to be finite,

$$\left|\frac{49}{100}\alpha^2\right| < 1, \quad \left|\frac{1}{25}\alpha^2\right| < 1, \text{ and } \left|\frac{7}{50}\alpha^2\right| < 1,$$

or equivalently,

$$|\alpha| < \frac{10}{7}, \quad |\alpha| < 5, \text{ and } |\alpha| < \sqrt{\frac{50}{7}}.$$

Hence, α should satisfy $|\alpha| < \frac{10}{7}$.

1. Consider the following two systems

$$y_1(n) = \frac{1}{4}\sum_{k=n-3}^{n} x(k),$$

$$y_2(n) = \begin{cases} x\left(\frac{n}{3}\right) & n = 3k, k \text{ integer} \\ 0 & n \neq 3k, k \text{ integer.} \end{cases}$$

(a) Determine whether the systems are linear, time-invariant, relaxed, BIBO stable, and causal. Justify your answer to receive full credit.

Solution:

Properties	$y_1(n)$	$y_2(n)$
Relaxed	Yes	No
Linear	Yes	Yes
Time-Invariant	Yes	No
BIBO Stable	Yes	Yes
Causal	Yes	No

- System $y_1(n)$:
  · $y_1(n) = \frac{1}{4}\sum_{k=n-3}^{n} x(k) = \frac{1}{4}[x(n-3) + x(n-2) + x(n-1) + x(n)]$ can be considered as a relaxed constant coefficient difference equation. It is relaxed, linear, time-invariant, BIBO stable, and causal.
- System $y_2(n)$:
  · Not relaxed: Counter example: if the first non-zero sample of $x(n)$ is at $n = -1$, then $y_2(-3)$ will be non-zero, which is before -1.
  · Linear: Let two outputs be

$$y_2^a(n) = \begin{cases} x_a\left(\frac{n}{3}\right) & n = 3k, k \text{ integer} \\ 0 & n \neq 3k, k \text{ integer.} \end{cases},$$

and

$$y_2^b(n) = \begin{cases} x_b\left(\frac{n}{3}\right) & n = 3k, k \text{ integer} \\ 0 & n \neq 3k, k \text{ integer.} \end{cases},$$

for an input $x_a(n)$ and $x_b(n)$, respectively. The output $y(n)$ for an input $Ax_a(n)+Bx_b(n)$ with some constants A and B is given by
i) if $n = 3k$, k integer,

$$\begin{aligned} y(n) &= Ax_a\left(\frac{n}{3}\right) + Bx_b\left(\frac{n}{3}\right) \\ &= Ay_2^a(n) + By_2^b(n). \end{aligned}$$

ii) if $n \neq 3k$, k integer,

$$y(n) = 0.$$

Based on i) and ii), $y(n)$ for an input $Ax_a(n) + Bx_b(n)$ is

$$y(n) = Ay_2^a(n) + By_2^b(n),$$

which implies that system $y_2(n)$ is linear.

- BIBO Stable: For a bounded sequence $|x(n)| \leq M < \infty$ with a finite positive number M,

$$|y_2(n)| = \begin{cases} |x\left(\frac{n}{3}\right)| \leq M < \infty & n = 3k, k \text{ integer} \\ 0 < \infty & n \neq 3k, k \text{ integer.} \end{cases},$$

implying that $|y_2(n)|$ is bounded output.
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- Not causal: For example, $y_2(-3)$ depends on $x(-1)$, which is on the future of $y_2(-3)$.

(b) Given $x(n) = nu(n)$, compute and plot $y_1(n)$ and $y_2(n)$ for $0 \leq n \leq 5$.

Solution:

n	0	1	2	3	4	5
$y_1(n)$	0	1/4	3/4	$6/4 = 3/2$	$10/4 = 5/2$	$14/4 = 7/2$
$y_2(n)$	0	0	0	1	0	0

The corresponding plots are shown in Figure 1.

(c) Find the z-transform of $y_1(n)$ and $y_2(n)$ in terms of the z-transform of $x(n)$.

Solution:
z-transform for $y_1(n) = \frac{1}{4}\sum_{k=n-3}^{n} x(k) = \frac{1}{4}[x(n-3) + x(n-2) + x(n-1) + x(n)]$ gives

$$\begin{aligned} Y_1(z) &= \frac{1}{4}[z^{-3}X(z) + z^{-2}X(z) + z^{-1}X(z) + X(z)] \\ &= \frac{z^{-3} + z^{-2} + z^{-1} + 1}{4}X(z). \end{aligned}$$

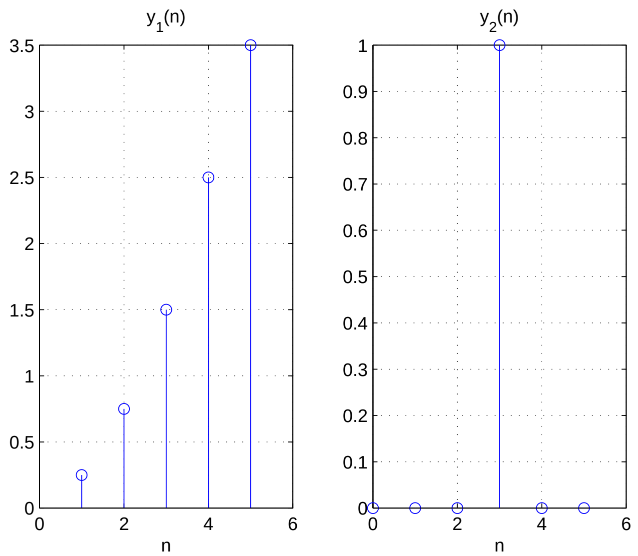
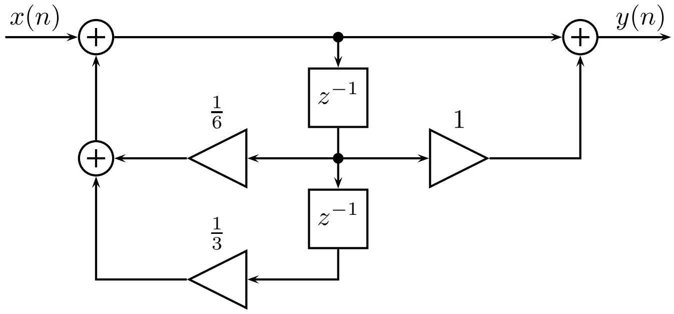


Figure 1: Plot for $y_1(n)$ and $y_2(n)$.

z-transform for $y_2(n)$ is given by

$$\begin{aligned} Y_2(z) &= \sum_{n=-\infty}^{\infty} y_2(n) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{3}\right) z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x(k) z^{-3k} \\ &= \sum_{k=-\infty}^{\infty} x(k) (z^3)^{-k} \\ &= X(z^3). \end{aligned}$$

2. A causal system is described by the following block diagram.



(a) Determine the constant coefficient difference equation that describes the system, and find its impulse response.

Solution:
Interchanging the order of the two systems in cascade, we get

$$y(n) = \frac{1}{6}y(n-1) + \frac{1}{3}y(n-2) + x(n) + x(n-1).$$

The impulse response can be found by computing the zero-state response to $x(n) = \delta(n)$. We first find the homogeneous solution:

$$\lambda^2 - \frac{1}{6}\lambda - \frac{1}{3} = 0 \quad \Rightarrow \quad \lambda_1 = -\frac{1}{2}, \quad \lambda_2 = \frac{2}{3}$$

and hence,

$$y_h(n) = C_1\left(-\frac{1}{2}\right)^n + C_2\left(\frac{2}{3}\right)^n, \quad \text{for all } n.$$

Letting $x(n) = \delta(n)$ and assuming that the system is relaxed, we get $y(0) = 1$ and $y(1) = \frac{1}{6}y(0) + 1 = \frac{7}{6}$. From the homogeneous solution we get

$$y(0) = C_1 + C_2 = 1$$
$$y(1) = -\frac{1}{2}C_1 + \frac{2}{3}C_2 = \frac{7}{6}$$

and solving for C_1 and C_2 yields $C_1 = -\frac{3}{7}$ and $C_2 = \frac{10}{7}$, and consequently,

$$h(n) = \left[-\frac{3}{7}\left(-\frac{1}{2}\right)^n + \frac{10}{7}\left(\frac{2}{3}\right)^n\right]u(n).$$

(b) Given $x(n) = n2^n u(-n)$, find the output of the system using the z-transform.

Solution:
The output $y(n)$ when $x(n) = n2^n u(-n)$ can be found by computing the inverse z-transform of $Y(z) = H(z)X(z)$. The z-transform of $h(n)$ is readily found from the impulse response:

$$H(z) = \frac{1 + z^{-1}}{1 - \frac{1}{6}z^{-1} - \frac{1}{3}z^{-2}}, \quad \text{ROC: } |z| > \frac{2}{3}.$$

The z-transform of $x(n)$ is given by

$$X(z) = \mathcal{Z}\{x(n)\} = -z\frac{d}{dz}\mathcal{Z}\{2^n u(-n)\} = -z\frac{d}{dz}\left[\sum_{n=-\infty}^{0} 2^n z^{-n}\right]$$
$$= -z\frac{d}{dz}\left[\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right] = -z\frac{d}{dz}\left[\frac{1}{1-\frac{1}{2}z}\right] = -\frac{2z}{(2-z)^2}, \quad \text{ROC: } |z| < 2.$$

Consequently, $Y(z)$ is given by

$$Y(z) = H(z)X(z) = \frac{-2z^3 - 2z^2}{(z+\frac{1}{2})(z-\frac{2}{3})(z-2)^2}, \quad \text{ROC: } \frac{2}{3} < |z| < 2.$$

Using partial fraction expansion, we get

$$\frac{Y(z)}{z} = \frac{-2z^2 - 2z}{(z+\frac{1}{2})(z-\frac{2}{3})(z-2)^2} = \frac{A}{z+\frac{1}{2}} + \frac{B}{z-\frac{2}{3}} + \frac{C}{z-2} + \frac{D}{(z-2)^2}$$

where

$$A = \left(z+\frac{1}{2}\right)\frac{Y(z)}{z}\bigg|_{z=-\frac{1}{2}} = -\frac{12}{175}$$
$$B = \left(z-\frac{2}{3}\right)\frac{Y(z)}{z}\bigg|_{z=\frac{2}{3}} = -\frac{15}{14}$$
$$C = \frac{d}{dz}\left[(z-2)^2\frac{Y(z)}{z}\right]\bigg|_{z=2} = \frac{57}{50}$$
$$D = (z-2)^2\frac{Y(z)}{z}\bigg|_{z=2} = -\frac{18}{5}.$$

Thus,

$$Y(z) = -\frac{12}{175}\frac{z}{z+\frac{1}{2}} - \frac{15}{14}\frac{z}{z-\frac{2}{3}} + \frac{57}{50}\frac{z}{z-2} - \frac{9}{5}\frac{2z}{(z-2)^2}$$

and since the ROC of $Y(z)$ is given by $\frac{2}{3} < |z| < 2$, we get

$$\frac{z}{z+\frac{1}{2}} \leftrightarrow \left(-\frac{1}{2}\right)^n u(n)$$
$$\frac{z}{z-\frac{2}{3}} \leftrightarrow \left(\frac{2}{3}\right)^n u(n)$$
$$\frac{z}{z-2} \leftrightarrow -2^n u(-n-1)$$
$$\frac{2z}{(z-2)^2} \leftrightarrow -n2^n u(-n-1)$$

and consequently,

$$y(n) = \left[-\frac{12}{175}\left(-\frac{1}{2}\right)^n - \frac{15}{14}\left(\frac{2}{3}\right)^n\right]u(n) + \left[-\frac{57}{50} + \frac{9}{5}n\right]2^n u(-n-1).$$

Alternatively, partial fraction expansion on $Y(z)$ yields

$$y(n) = \left[\frac{6}{175}\left(-\frac{1}{2}\right)^{n-1} - \frac{5}{7}\left(\frac{2}{3}\right)^{n-1}\right]u(n-1) + \left[-\frac{57}{25} + \frac{18}{5}n\right]2^{n-1}u(-n).$$

3. The difference equation of a relaxed system is:

$$y(n) + 0.5y(n-1) - 0.14y(n-2) = x(n)$$

(a) Find a closed form for the impulse response $h(n)$ (i.e., the zero state system output when the input is an impulse).

Solution:
By setting $x(n) = \delta(n)$, we have

$$h(n) + 0.5h(n-1) - 0.14h(n-2) = \delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases}$$

Hence, for $n \geq 1$, $h(n)$ can be found by solving homogeneous difference equation

$$h(n) + 0.5h(n-1) - 0.14h(n-2) = 0. \tag{1}$$

The characteristic polynomial for (1) can be solved by

$$\left(\lambda + \frac{7}{10}\right)\left(\lambda - \frac{1}{5}\right) = 0,$$

thus, the modes are

$$\lambda_1 = -\frac{7}{10}, \quad \lambda_2 = \frac{1}{5}.$$

Hence,

$$h(n) = C_1\left(-\frac{7}{10}\right)^n + C_2\left(\frac{1}{5}\right)^n, \quad n \geq 0.$$

Since the system is relaxed, $y(-1) = h(-1) = 0$ and $h(0) = \delta(0) = 1$, which gives

$$C_1 = \frac{7}{9}, \quad C_2 = \frac{2}{9}.$$

Therefore,

$$h(n) = \left[\frac{7}{9}\left(-\frac{7}{10}\right)^n + \frac{2}{9}\left(\frac{1}{5}\right)^n\right]u(n).$$

(b) If the input to the system is $x(n) = 2\delta(n) + \delta(n-2)$, what is the output?

Solution:
The output $y(n)$ of the system can be expressed as the convolution of $x(n)$ and $h(n)$, i.e.,

$$\begin{aligned} y(n) &= x(n) * h(n) \\ &= [2\delta(n) + \delta(n-2)] * h(n) \\ &= 2h(n) + h(n-2). \end{aligned}$$

Hence, using $h(n)$ in part (a) gives us
i) $n \geq 2$,

$$\begin{aligned}
y(n) &= 2h(n) + h(n-2) \\
&= 2\left[\frac{7}{9}\left(-\frac{7}{10}\right)^n + \frac{2}{9}\left(\frac{1}{5}\right)^n\right]u(n) + \left[\frac{7}{9}\left(-\frac{7}{10}\right)^{n-2} + \frac{2}{9}\left(\frac{1}{5}\right)^{n-2}\right]u(n-2) \\
&= \left[2\cdot\frac{7}{9} + \frac{7}{9}\cdot\left(\frac{7}{10}\right)^{-2}\right]\left(-\frac{7}{10}\right)^n + \left[2\cdot\frac{2}{9} + \frac{2}{9}\cdot\left(\frac{1}{5}\right)^{-2}\right]\left(\frac{1}{5}\right)^n \\
&= \frac{22}{7}\left(-\frac{7}{10}\right)^n + 6\left(\frac{1}{5}\right)^n.
\end{aligned}$$

ii) $0 \leq n \leq 1$,

$$\begin{aligned}
y(n) &= 2h(n) \\
&= 2\left[\frac{7}{9}\left(-\frac{7}{10}\right)^n + \frac{2}{9}\left(\frac{1}{5}\right)^n\right]u(n) \\
&= \frac{14}{9}\left(-\frac{7}{10}\right)^n + \frac{4}{9}\left(\frac{1}{5}\right)^n.
\end{aligned}$$

Therefore,

$$y(n) = \begin{cases} 0, & n < 0, \\ \frac{14}{9}\left(-\frac{7}{10}\right)^n + \frac{4}{9}\left(\frac{1}{5}\right)^n, & n = 0, 1, \\ \frac{22}{7}\left(-\frac{7}{10}\right)^n + 6\left(\frac{1}{5}\right)^n, & n \geq 2. \end{cases}$$

(c) For what values of α is $g(n) = \alpha^n h(n)$ a finite energy sequence?

Solution:
The energy E_g of $g(n)$ is given by

$$\begin{aligned}
E_g &= \sum_{n=-\infty}^{\infty} |g(n)|^2 \\
&= \sum_{n=-\infty}^{\infty} |\alpha^n h(n)|^2 \\
&= \sum_{n=0}^{\infty}\left|\alpha^n\left\{\frac{7}{9}\left(-\frac{7}{10}\right)^n + \frac{2}{9}\left(\frac{1}{5}\right)^n\right\}\right|^2 \\
&= \sum_{n=0}^{\infty}\left|\frac{7}{9}\left(-\frac{7}{10}\alpha\right)^n + \frac{2}{9}\left(\frac{1}{5}\alpha\right)^n\right|^2 \\
&= \sum_{n=0}^{\infty}\frac{49}{81}\left(\frac{49}{100}\alpha^2\right)^n + \sum_{n=0}^{\infty}\frac{4}{81}\left(\frac{1}{25}\alpha^2\right)^n + \sum_{n=0}^{\infty}\frac{28}{81}\left(-\frac{7}{50}\alpha^2\right)^n.
\end{aligned}$$

Thus, for the energy to be finite,

$$\left|\frac{49}{100}\alpha^2\right| < 1, \quad \left|\frac{1}{25}\alpha^2\right| < 1, \text{ and } \left|\frac{7}{50}\alpha^2\right| < 1,$$

or equivalently,

$$|\alpha| < \frac{10}{7}, \quad |\alpha| < 5, \text{ and } |\alpha| < \sqrt{\frac{50}{7}}.$$

Hence, α should satisfy $|\alpha| < \frac{10}{7}$.