**EE113: Digital Signal Processing**
Prof. Mihaela van der Schaar — Nov. 9, 2006

# MIDTERM EXAMINATION

(Closed Book - 50 pts)

1. (22 pts) Consider the following two systems

$$y_1(n) = \frac{1}{3} \sum_{k=n-2}^{n} x(k) \tag{1}$$

$$y_2(n) = \begin{cases} x\left(\frac{n}{2}\right), & \text{when n is even;} \\ 0, & \text{when n is odd.} \end{cases} \tag{2}$$

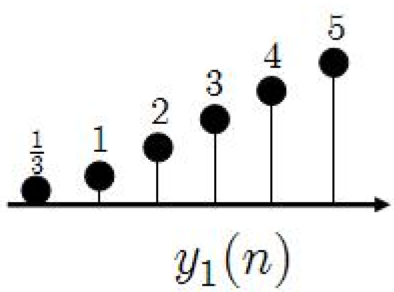
   (a) (10 pts) Determine whether the systems are linear, time-invariant, relaxed, BIBO stable, and causal. Clearly state your justifications to receive full credit.

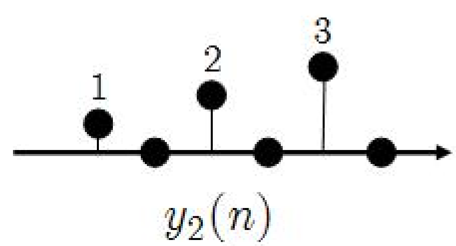
   (b) (6 pts) Given $x(n) = (n+1)u(n)$, compute and plot $y_1(n)$ and $y_2(n)$ for $0 \le n \le 5$.

   (c) (6 pts) Find the Z-transform of $y_1(n)$ and $y_2(n)$ in terms of $X(z)$.

(a) answer: The first system is LTI. Notice that $y(n) = h(n) * x(n)$, where $h(n) = \frac{1}{3}(\delta(n) + \delta(n-1) + \delta(n-2))$.
The system is relaxed since $y(n) = 0$ when $x(n) = 0$.
The system is BIBO stable since if $|x(n)| < M$ for all $n$, $|y(n)| = |\frac{1}{3}(x(n)+x(n-1)+x(n-2))| < \frac{1}{3}(M+M+M) = 3M$.
The system is causal since $y(n)$ depends only on current and past inputs of $x(n)$.

The second system is linear. For $\alpha x_a(n) + \beta x_b(n)$, the odd samples will still be mapped to 0, while the even samples will be $y(n) = \alpha x_a(\frac{n}{2}) + \beta x_b(\frac{n}{2}) = \alpha y_a(n) + \beta y_b(n)$.
The system is NOT time invariant. Note that $y_1(n) = S[x(n-1)]$ returns $x(\frac{n-1}{2})$ when n is odd, but $y(n-1) = 0$ when n is odd.
The system is not relaxed. If $x(-1) = 1$, then $y(-2) = 1$.
The system is BIBO stable. $|y(n)| = |x(n/2)| < M$.
The system is not causal. If $x(n) = u(n+1)$, $y(-2) = 1$, so $y(n)$ becomes non-zero before $x(n)$ becomes non-zero.




(b) answer:

(c) answer:

$$Y_1(z) = \frac{1}{3}(X_1(z) + z^{-1}X_1(z) + z^{-2}X_1(z))$$

$$\begin{aligned}
Y_2(z) &= \sum_{n=-\infty}^{\infty} y_2(n) z^{-n} \\
&= \sum_{k=-\infty}^{\infty} x_2(k) z^{-2k} \\
&= \sum_{k=-\infty}^{\infty} x_2(k) (z^2)^{-k} \\
&= X(z^2)
\end{aligned}$$

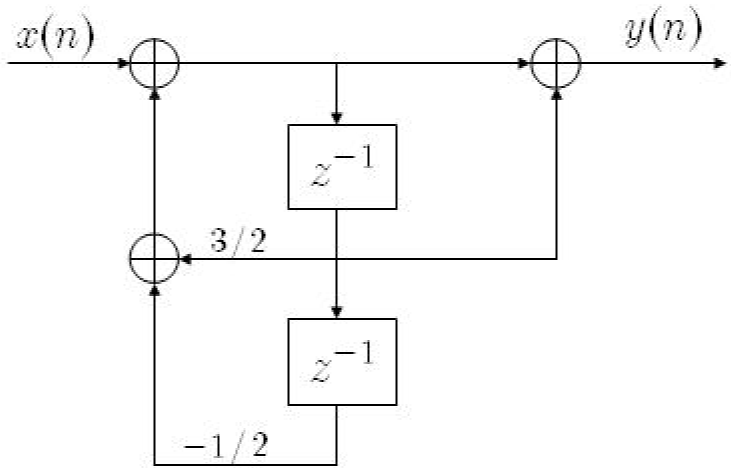



Figure 1: System block diagram.

2. (16 pts) A relaxed causal system is described by the block diagram above.

(a) (6 pts) Determine the constant coefficient difference equation that describes this system, and find its impulse response.

(b) (6 pts) given $x(n) = n2^n u(-n)$, find the output of the system using Z-transform.

(c) (4 pts) Determine the energy of the output sequence.

(a) answer: $y(n) - \frac{3}{2}y(n-1) + \frac{1}{2}y(n-2) = x(n) + x(n-1)$.
There are two ways to find $h(n)$, either by finding the transfer function or by evaluating directly with $\delta(n)$ as the input.

The transfer function method:

$$\begin{aligned}
Y(z) - \frac{3}{2}z^{-1}Y(z) + \frac{1}{2}z^{-2}Y(z) &= X(z) + z^{-1}X(z) \\
Y(z)(1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}) &= X(z)(1 + z^{-1}) \\
H(z) = Y(z)/X(z) &= \frac{1 + z^{-1}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} \\
&= z\left(\frac{z+1}{z^2 - \frac{3}{2}z + \frac{1}{2}}\right) \\
&= z\left(\frac{4}{z-1} + \frac{-3}{z - \frac{1}{2}}\right)
\end{aligned}$$

The inverse transform for $H(z)$ is $h(n) = 4u(n) - 3\left(\frac{1}{2}\right)^n u(n)$.

Solving it using homogeneous equations would require first the modes $\lambda = 1$ and $\lambda = 1/2$.

Using initial conditions at $n = 0$ and $n = 1$, since the equation becomes homogeneous at $n = 2$, we get:

$$\begin{aligned} y(0) = \delta(0) = 1 &= C_1 + C_2 \\ y(1) = \frac{3}{2}y(0) + \delta(0) = \frac{5}{2} &= C_1 + \frac{1}{2}C_2 \end{aligned}$$

We end up with $C_1 = 4$ and $C_2 = -3$, which agrees with the transfer function method.

(b) answer: $x(n) = n2^n u(-n) = n2^n u(-n-1)$ since $x(0) = 0$. Hence, the Z-transform is:

$$X(z) = \frac{-2z}{(z-2)^2} \text{for} |z| < 2$$

So $Y(z) = H(z)X(z) = \frac{-2z(z+1)}{(z-1)(z-\frac{1}{2})(z-2)^2}$ without pole-zero cancellation, hence the ROC is the intersection of the ROCs of $H(z)$ and $X(z)$, which is $1 < |z| < 2$. Removing a factor of $z$ from the numerator and using partial fractions, we get:

$$Y(z) = -\frac{8z}{z-1} + \frac{8z}{3(z-\frac{1}{2})} + \frac{4z}{3(z-2)} - \frac{8z}{(z-2)^2}$$

Given our region of convergence, this gives us:

$$y(n) = -8u(n) + \frac{8}{3}\left(\frac{1}{2}\right)^n u(n) - \frac{4}{3}2^n u(-n-1) + 4n2^n u(-n-1)$$

(c) answer: The answer is $\infty$. To see this, simply notice that there is a non-zero coefficient in front of the $u(n)$ term. All other terms go to 0 as $n$ goes to $\infty$, hence the sequence will approach 8 as $n \to \infty$.

Although not necessary for the answer to this problem, we show a way to verify this using z-transforms. Let us define $w(n) = y^2(n)$. Since $y(n)$ is a real sequence, $\sum_{n=-\infty}^{\infty} w(n) = \varepsilon_y = W(1)$.

$$w(n) = \left[\frac{64}{9}\left(\frac{1}{4}\right)^n - \frac{128}{3}\left(\frac{1}{2}\right)^n + 64\right]u(n) + \left[\frac{16}{9}4^n - \frac{32}{3}n4^n + 16n^2 4^n\right]u(-n-1)$$

Using z-transform properties, we get:

$$W(z) = \frac{64z}{9(z-\frac{1}{4})} - \frac{128z}{3(z-\frac{1}{2})} + \frac{64z}{z-1} - \frac{16}{9(z-4)} + \frac{128z}{3(z-4)^2} - z\frac{-16z-64}{(z-4)^3}$$

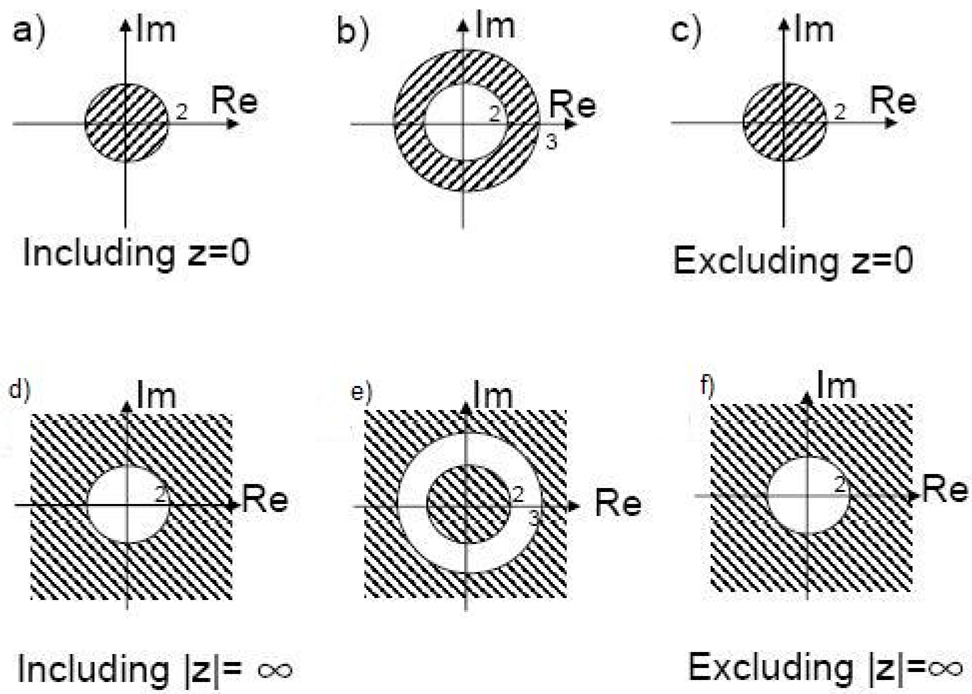
The term with $z - 1$ in the denominator will go to $\infty$ when $z = 1$. Hence, the energy is $\infty$.

3. (12 pts) Given the plots (a-f), determine which one corresponds to each of the sequences below.

(a) (4 pts) $x_1(n) = a2^n u(-n-10) + b3^{n-3}u(-n+100)$

(b) (4 pts) $x_1(n) = a3^n u(-n-10) + b2^{n-3}u(n+100)$

(c) (4 pts) $x_1(n) = a2^n u(n-10) + bu(n+100)$




(a) answer: c. Because the sequence includes a finite number of non-zero terms for positive values of $n$, the z-transform does not converge at $z = 0$.

(b) answer: b, $2 < |z| < 3$. The sequence is just a linear combination of shifted versions of $3^n u(-n-1)$ and $2^n u(n)$.

(c) answer: f, $2 < |z| < \infty$. This is because there are non-zero terms where $n < 0$.