

1. Problem 1

- (a) Since $y(n) = x^2(n)$, we have $Y(e^{j\omega}) = X(e^{j\omega}) * X(e^{j\omega})$. Hence, $Y(e^{j\omega})$ will occupy twice the frequency band that $X(e^{j\omega})$ does. To get $y_c(n) = x_c^2(n)$, the highest frequency of $y_c(n)$ is $0.5/T = 5000$ Hz. Consequently, the highest frequency of $x_c(n)$ is 2500 Hz, or $\pi \times 5000$ rads/second.

- (b) The magnitude response is

$$|H(e^{j\omega})| = \frac{\sqrt{2 - 2 \cos(\omega - \omega_0)} \cdot \sqrt{2 - 2 \cos(\omega + \omega_0)}}{\sqrt{1.81 - 1.8 \cos(\omega - \omega_0)} \cdot \sqrt{1.81 - 1.8 \cos(\omega + \omega_0)}}.$$

The phase response is

$$\begin{aligned} \angle H(e^{j\omega}) &= \tan^{-1} \left(\frac{\sin(\omega - \omega_0)}{1 - \cos(\omega - \omega_0)} \right) + \tan^{-1} \left(\frac{\sin(\omega + \omega_0)}{1 - \cos(\omega + \omega_0)} \right) \\ &\quad - \tan^{-1} \left(\frac{0.9 \sin(\omega - \omega_0)}{1 - 0.9 \cos(\omega - \omega_0)} \right) - \tan^{-1} \left(\frac{0.9 \sin(\omega + \omega_0)}{1 - 0.9 \cos(\omega + \omega_0)} \right). \end{aligned}$$

- (c) Since $f = 60$ Hz in frequency domain corresponds to $\Omega = 2\pi f = 2\pi \times 60$ rads/second in angular frequency domain, we have

$$\omega = \Omega T = 2\pi \times 60 \times 10^{-4} = \frac{3\pi}{250}$$

in the transform domain of DTFT.

To filter the 60Hz out, we need

$$|H(e^{j\omega})|_{\omega=\frac{3\pi}{250}} = 0,$$

which holds true if

$$\cos(\omega - \omega_0)|_{\omega=\frac{3\pi}{250}} = 1.$$

Hence, we have $\omega_0 = \frac{3\pi}{250}$.

2. Problem 2

- (a) $H_1(e^{jw})$ corresponds to a frequency shifted version of $H_{lp}(e^{j\omega})$, specifically:

$$H_1(e^{jw}) = H_{lp}(e^{j(w-\pi)})$$

Thus, we have

$$H_1(e^{jw}) = \begin{cases} 0, & |w| \leq 0.8\pi \\ 1, & 0.8\pi < |w| \leq \pi \end{cases}$$

This is a highpass filter.

- (b) $H_2(e^{jw})$ corresponds to a frequency modulated version of $H_{lp}(e^{j\omega})$, specifically:

$$H_2(e^{jw}) = H_{lp}(e^{jw}) * (\delta(w - 0.5\pi) + \delta(w + 0.5\pi)), \quad |w| \leq \pi$$

Thus, we have

$$H_2(e^{jw}) = \begin{cases} 0, & |w| \leq 0.3\pi \\ 1, & 0.3\pi < |w| < 0.7\pi \\ 0, & 0.7\pi \leq |w| \leq \pi \end{cases}$$

This is a bandpass filter.

- (c) $H_3(e^{jw})$ corresponds to a periodic convolution of $H_{lp}(e^{j\omega})$ with another lowpass filter, specifically:

$$H_3(e^{jw}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\theta}) H_{lp}(e^{j(w-\theta)}) d\theta$$

where $H(e^{jw})$ is given by

$$H(e^{jw}) = \begin{cases} 1, & |w| < 0.1\pi \\ 0, & 0.1\pi \leq |w| \leq \pi \end{cases}$$

Carrying out the convolution, we get

$$H_3(e^{jw}) = \begin{cases} 0.1, & |w| < 0.1\pi \\ -\frac{|w|}{2\pi} + 0.15, & 0.1\pi \leq |w| \leq 0.3\pi \\ 0, & 0.3\pi < |w| \leq \pi \end{cases}$$

This is lowpass filter.

(d) The energy of $h_3(n)$ is computed using Parseval theorem

$$\begin{aligned} E &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_3(e^{jw})|^2 dw \\ &= \frac{1}{2\pi} \left(\int_{-0.1\pi}^{0.1\pi} (0.1)^2 + 2 \left(\int_{0.1\pi}^{0.3\pi} \left(0.15^2 - \frac{0.3}{2\pi} w + \frac{1}{4\pi^2} w^2 \right) dw \right) \right) \end{aligned}$$

The answer is considered to be complete up to the above equality. Further calculation gives us the energy of $h_3(n)$ as 0.005.

3. Problem 3. Let $h(n) = \delta(n) - \frac{1}{2}\delta(n - n_0)$

(a) The Z-transform of $h(n)$

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h(n)z^{-n} \\ &= 1 - \frac{1}{2}z^{-n_0} \end{aligned}$$

The N -point DFT of $h(n)$ with $N = 4n_0$ is

$$\begin{aligned} H(k) &= \sum_{n=0}^{4n_0-1} h(n)e^{-j\frac{2\pi kn}{4n_0}}, \quad 0 \leq k \leq 4n_0 - 1 \\ &= 1 - \frac{1}{2}e^{-j(\pi/2)k} \end{aligned}$$

(b) The impulse response of $h_i(n)$ is

$$\begin{aligned} H_i(z) &= \frac{1}{1 - 1/2z^{-n_0}}, \quad |z| > \left(\frac{1}{2}\right)^{-n_0} \\ h_i(n) &= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{n/n_0} \delta(n - kn_0) \end{aligned}$$

This is IIR filter, so infinite duration.

(c)

$$G(k) = \frac{1}{H(k)} = \frac{1}{1 - \frac{1}{2}e^{-j(\pi/2)k}}, \quad 0 \leq k \leq 4n_0 - 1$$

The impulse response, $g(n)$ is just $h_i(n)$ time-aliased by $4n_0$ points

$$\begin{aligned} g(n) &= \left(1 + \frac{1}{16} + \frac{1}{256} + \dots\right)\delta(n) + \left(\frac{1}{2} + \frac{1}{32} + \frac{1}{512} + \dots\right)\delta(n - n_0) \\ &\quad + \left(\frac{1}{4} + \frac{1}{64} + \frac{1}{1024} + \dots\right)\delta(n - 2n_0) + \left(\frac{1}{8} + \frac{1}{128} + \frac{1}{2048} + \dots\right)\delta(n - 3n_0) \\ &= \frac{16}{15}\delta(n) + \frac{8}{15}\delta(n - n_0) + \frac{4}{15}\delta(n - 2n_0) + \frac{2}{15}\delta(n - 3n_0) \end{aligned}$$

(d)

$$\begin{aligned} y(n) = g(n) * h(n) &= \frac{16}{15}\delta(n) + \frac{8}{15}\delta(n - n_0) + \frac{4}{15}\delta(n - 2n_0) + \frac{2}{15}\delta(n - 3n_0) \\ &\quad - \frac{8}{15}\delta(n - n_0) - \frac{4}{15}\delta(n - 2n_0) - \frac{2}{15}\delta(n - 3n_0) - \frac{1}{15}\delta(n - 4n_0) \\ &= \frac{16}{15}\delta(n) - \frac{1}{15}\delta(n - 4n_0) \end{aligned}$$

Since $y(n) \neq \delta(n)$, we cannot perfectly recover $x(n)$ from $y(n)$ using $g(n)$.

Indeed,

$$G(k)H(k) = 1, \quad 0 \leq k \leq 4n_0 - 1$$

However, this relationship is only true at $4n_0$ distinct frequencies.

This fact does not imply that for all frequencies w :

$$G(e^{jw})H(e^{jw}) = 1$$

4. Problem 4 Let $x(n) = 0, n < 0, n > 7$ be a real eight-point sequence and let $X(k)$ be its eight-point DFT

(a)

$$\frac{1}{8} \sum_{k=0}^7 X(k) e^{j(2\pi/8)k \times 9} = \frac{1}{8} \sum_{k=0}^7 X(k) e^{j(2\pi/8)k \times 1} = x(1)$$

(b)

$$\begin{aligned} V(k) &= X(z) \Big|_{z=2e^{j(2\pi k+\pi)/8}} \\ &= \sum_{n=-\infty}^{\infty} x(n) z^{-n} \Big|_{z=2e^{j(2\pi k+\pi)/8}} \\ &= \sum_{n=0}^7 x(n) z^{-n} \Big|_{z=2e^{j(2\pi k+\pi)/8}} \\ &= \sum_{n=0}^7 x(n) (2e^{j\pi/8})^{-n} e^{-j\frac{2\pi k}{8}n} \\ &= \sum_{n=0}^7 v(n) e^{-j\frac{2\pi k}{8}n} \end{aligned}$$

Thus, we have

$$v(n) = x(n)(2e^{j\pi/8})^{-n}$$

(c)

$$\begin{aligned} w(n) &= \frac{1}{4} \sum_{k=0}^3 W(k) e^{j\frac{2\pi kn}{4}} \\ &= \frac{1}{4} \sum_{k=0}^3 (X(k) + X(k+4)) e^{j\frac{2\pi}{4}kn} \\ &= \frac{1}{4} \sum_{k=0}^3 X(k) e^{j\frac{2\pi}{4}kn} + \frac{1}{4} \sum_{k=0}^3 X(k+4) e^{j\frac{2\pi}{4}kn} \\ &= \frac{1}{4} \sum_{k=0}^3 X(k) e^{j\frac{2\pi}{4}kn} + \frac{1}{4} \sum_{k=4}^7 X(k) e^{j\frac{2\pi}{4}kn} \\ &= \frac{1}{4} \sum_{k=0}^7 X(k) e^{j\frac{2\pi}{4}k2n} \\ &= 2x(2n) \end{aligned}$$

Thus,

$$w(n) = 2x(2n)$$

(d) Note that $Y(k)$ can be written as

$$\begin{aligned} Y(k) &= X(k) + (-1)^k X(k) \\ &= X(k) + e^{-j\frac{2\pi k^4}{8}} X(k) \end{aligned}$$

Using DFT properties, we have

$$y(n) = x(n) + x((n-4)\bmod 8)$$

1. Problem 1 Let $x(n) = 0$, $n < 0$, $n > 7$ be a real eight-point sequence and let $X(k)$ be its eight-point DFT

(a)

$$\frac{1}{8} \sum_{k=0}^7 X(k) e^{j(2\pi/8)k \times 9} = \frac{1}{8} \sum_{k=0}^7 X(k) e^{j(2\pi/8)k \times 1} = x(1)$$

(b)

$$\begin{aligned} V(k) &= X(z) \Big|_{z=2e^{j(2\pi k+\pi)/8}} \\ &= \sum_{n=-\infty}^{n=\infty} x(n) z^{-n} \Big|_{z=2e^{j(2\pi k+\pi)/8}} \\ &= \sum_{n=0}^{n=7} x(n) z^{-n} \Big|_{z=2e^{j(2\pi k+\pi)/8}} \\ &= \sum_{n=0}^{n=7} x(n) (2e^{j\pi/8})^{-n} e^{-j\frac{2\pi k}{8}n} \\ &= \sum_{n=0}^{n=7} v(n) e^{-j\frac{2\pi k}{8}n} \end{aligned}$$

Thus, we have

$$v(n) = x(n) (2e^{j\pi/8})^{-n}$$

(c)

$$\begin{aligned}w(n) &= \frac{1}{4} \sum_{k=0}^3 W(k) e^{j \frac{2\pi kn}{4}} \\&= \frac{1}{4} \sum_{k=0}^3 (X(k) + X(k+4)) e^{j \frac{2\pi}{4} kn} \\&= \frac{1}{4} \sum_{k=0}^3 X(k) e^{j \frac{2\pi}{4} kn} + \frac{1}{4} \sum_{k=0}^3 X(k+4) e^{j \frac{2\pi}{4} kn} \\&= \frac{1}{4} \sum_{k=0}^3 X(k) e^{j \frac{2\pi}{4} kn} + \frac{1}{4} \sum_{k=4}^7 X(k) e^{j \frac{2\pi}{4} kn} \\&= \frac{1}{4} \sum_{k=0}^7 X(k) e^{j \frac{2\pi}{4} k2n} \\&= 2x(2n)\end{aligned}$$

Thus,

$$w(n) = 2x(2n)$$

(d) Note that $Y(k)$ can be written as

$$\begin{aligned}Y(k) &= X(k) + (-1)^k X(k) \\&= X(k) + e^{-j \frac{2\pi k4}{8}} X(k)\end{aligned}$$

Using DFT properties, we have

$$y(n) = x(n) + x((n-4) \bmod 8)$$

2. Problem 2. Let $h(n) = \delta(n) - \frac{1}{2}\delta(n - n_0)$

(a) The Z-transform of $h(n)$

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h(n)z^{-n} \\ &= 1 - \frac{1}{2}z^{-n_0} \end{aligned}$$

The N -point DFT of $h(n)$ with $N = 8n_0$ is

$$\begin{aligned} H(k) &= \sum_{n=0}^{8n_0-1} h(n)e^{-j\frac{2\pi kn}{8n_0}}, \quad 0 \leq k \leq 8n_0 - 1 \\ &= 1 - \frac{1}{2}e^{-j(\pi/4)k} \end{aligned}$$

(b) The impulse response of $h_i(n)$ is

$$\begin{aligned} H_i(z) &= \frac{1}{1 - 1/2z^{-n_0}}, \quad |z| > \left(\frac{1}{2}\right)^{-n_0} \quad \text{for causality} \\ h_i(n) &= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{kn_0} \delta(n - kn_0) \end{aligned}$$

This is IIR filter, so infinite duration.

(c)

$$G(k) = \frac{1}{H(k)} = \frac{1}{1 - \frac{1}{2}e^{-j(\pi/4)k}}, \quad 0 \leq k \leq 8n_0 - 1$$

The impulse response, $g(n)$ is just $h_i(n)$ time-aliased by $8n_0$ points

$$\begin{aligned} g(n) &= \left(1 + \frac{1}{2^8} + \frac{1}{2^{16}} + \dots\right)\delta(n) + \left(\frac{1}{2} + \frac{1}{2^9} + \frac{1}{2^{17}} + \dots\right)\delta(n - n_0) \\ &\quad + \dots \\ &\quad + \left(\frac{1}{2^7} + \frac{1}{2^{15}} + \frac{1}{2^{23}} + \dots\right)\delta(n - 7n_0) \\ &= \frac{256}{255}\delta(n) + \frac{128}{255}\delta(n - n_0) + \dots + \frac{2}{255}\delta(n - 7n_0) \end{aligned}$$

(d)

$$\begin{aligned} y(n) = g(n) * h(n) &= \frac{256}{255}\delta(n) + \dots + \frac{2}{255}\delta(n - 7n_0) \\ &\quad - \frac{128}{255}\delta(n - n_0) - \dots - \frac{1}{255}\delta(n - 8n_0) \\ &= \frac{256}{255}\delta(n) - \frac{1}{255}\delta(n - 8n_0) \end{aligned}$$

Since $y(n) \neq \delta(n)$, we cannot perfectly recover $x(n)$ from $y(n)$ using $g(n)$.

Indeed,

$$G(k)H(k) = 1, \quad 0 \leq k \leq 8n_0 - 1$$

However, this relationship is only true at $8n_0$ distinct frequencies.

This fact does not imply that for all frequencies w :

$$G(e^{jw})H(e^{jw}) = 1$$

3. Problem 3 An ideal low-pass filter has impulse response $h_{lp}(n)$ and frequency response

$$H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\omega| < 0.1\pi, \\ 0, & 0.1\pi \leq |\omega| \leq \pi. \end{cases}$$

- (a) $H_1(e^{jw})$ corresponds to a frequency shifted version of $H_{lp}(e^{j\omega})$, specifically:

$$H_1(e^{jw}) = H_{lp}(e^{j(w-\pi)})$$

Thus, we have

$$H_1(e^{jw}) = \begin{cases} 0, & |w| \leq 0.9\pi \\ 1, & 0.9\pi < |w| \leq \pi \end{cases}$$

This is a highpass filter.

- (b) $H_2(e^{jw})$ corresponds to a frequency modulated version of $H_{lp}(e^{j\omega})$, specifically:

$$H_2(e^{jw}) = \frac{1}{2}H_{lp}(e^{jw}) * (\delta(w - 0.5\pi) + \delta(w + 0.5\pi)), \quad |w| \leq \pi$$

Thus, we have

$$H_2(e^{jw}) = \begin{cases} 0, & |w| \leq 0.4\pi \\ .5, & 0.4\pi < |w| < 0.6\pi \\ 0, & 0.6\pi \leq |w| \leq \pi \end{cases}$$

This is a bandpass filter.

- (c) $H_3(e^{jw})$ corresponds to a periodic convolution of $H_{lp}(e^{j\omega})$ with another lowpass filter, specifically:

$$H_3(e^{jw}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\theta})H_{lp}(e^{j(w-\theta)})d\theta$$

where $H(e^{jw})$ is given by

$$H(e^{jw}) = \begin{cases} 1, & |w| < 0.1\pi \\ 0, & 0.1\pi \leq |w| \leq \pi \end{cases}$$

Carrying out the convolution, we get

$$H_3(e^{jw}) = \begin{cases} -\frac{|w|}{2\pi} + 0.1, & |w| \leq 0.2\pi \\ 0, & 0.2\pi < |w| \leq \pi \end{cases} \quad (1)$$

This is lowpass filter.

(d) The energy of $h_3(n)$ is computed using Parseval theorem

$$\begin{aligned} E &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_3(e^{jw})|^2 dw \\ &= \frac{1}{2\pi} 2 \left(\int_0^{0.2\pi} \left(0.1^2 - \frac{0.2}{2\pi} w + \frac{1}{4\pi^2} w^2 \right) dw \right) \end{aligned}$$

The answer is considered to be complete up to the above equality. Further calculation gives us the energy of $h_3(n)$ as 0.0007.

4. Problem 4

- (a) Since $y(n) = x^2(n)$, we have $Y(e^{j\omega}) = X(e^{j\omega}) * X(e^{j\omega})$. Hence, $Y(e^{j\omega})$ will occupy twice the frequency band that $X(e^{j\omega})$ does. To get $y_c(n) = x_c^2(n)$, the highest frequency of $y_c(n)$ is $0.5/T = 500$ Hz. Consequently, the highest frequency of $x_c(n)$ is 250 Hz, or $\pi \times 500$ rads/second.
- (b) The magnitude response is

$$|H(e^{j\omega})| = \frac{\sqrt{2 - 2 \cos(\omega - \omega_0)} \cdot \sqrt{2 - 2 \cos(\omega + \omega_0)}}{\sqrt{1.25 - \cos(\omega - \omega_0)} \cdot \sqrt{1.25 - \cos(\omega + \omega_0)}}.$$

The phase response is

$$\begin{aligned} \angle H(e^{j\omega}) &= \tan^{-1} \left(\frac{\sin(\omega - \omega_0)}{1 - \cos(\omega - \omega_0)} \right) + \tan^{-1} \left(\frac{\sin(\omega + \omega_0)}{1 - \cos(\omega + \omega_0)} \right) \\ &\quad - \tan^{-1} \left(\frac{0.5 \sin(\omega - \omega_0)}{1 - 0.5 \cos(\omega - \omega_0)} \right) - \tan^{-1} \left(\frac{0.5 \sin(\omega + \omega_0)}{1 - 0.5 \cos(\omega + \omega_0)} \right). \end{aligned}$$

- (c) Since $f = 100$ Hz in frequency domain corresponds to $\Omega = 2\pi f = 2\pi \times 100$ rads/second in angular frequency domain, we have

$$\omega = \Omega T = 2\pi \times 100 \times 10^{-3} = \frac{\pi}{5}$$

in the transform domain of DTFT.

To filter the 100Hz out, we need

$$|H(e^{j\omega})|_{\omega=\frac{\pi}{5}} = 0,$$

which holds true if

$$\cos(\omega - \omega_0)|_{\omega=\frac{\pi}{5}} = 1.$$

Hence, we have $\omega_0 = \frac{\pi}{5}$.