Problem 1. (15 pts) A system for the discrete-time spectral analysis of continuous-time signals is shown in Figure 1.

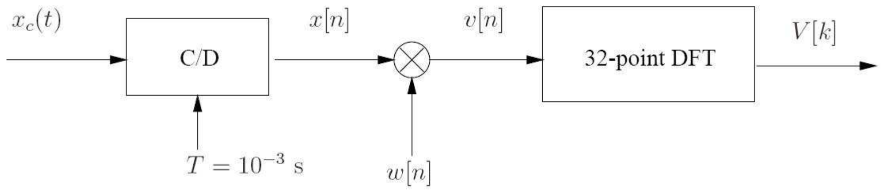

Figure 1: Spectral analysis system.

$\omega[n]$ is a rectangular window of length 32:

$$\omega[n] = \begin{cases} \frac{1}{32}, & 0 \leq n \leq 31 \\ 0, & \text{otherwise.} \end{cases}$$

Listed below are five signals, at least one of which was the input $x_c(t)$. Indicate which signal(s) could have been the input $x_c(t)$ which produced the plot of $|V[k]|$ shown in dB units in Figure 2. Provide reasoning for your choice(s).

$$x_1(t) = 1000cos(230\pi t) \quad x_2(t) = 1000cos(115\pi t)$$
$$x_3(t) = 10e^{j(460)\pi t} \quad x_4(t) = 1000e^{j(230)\pi t}$$
$$x_5(t) = 1000e^{j(250)\pi t}$$

Solution:

– $x_c(t)$ cannot be a cosine, because a cosine would have a second peak in the negative frequencies, i.e., in the upper half of the DFT. Thus, $x_1(t)$ and $x_2(t)$ are eliminated.

– $|V[4]| \approx 60\text{dB} \approx 1000$. $x_3(t)$ is eliminated because their amplitudes are too low to have produced a peak magnitude of 1000 in the DFT.

– The CT frequency of $x_c(t)$ cannot correspond exactly to a frequency $\omega_k = \frac{2\pi k}{32}$ sampled by the DFT. Otherwise, the DFT would be non-zero at exactly one value of k. Thus, $x_5(t)$ $(250\pi \rightarrow \frac{\pi}{4} \rightarrow k = 4)$ is eliminated.

– $x_4(t)$ is the only signal that could have been the input $x_c(t)$.

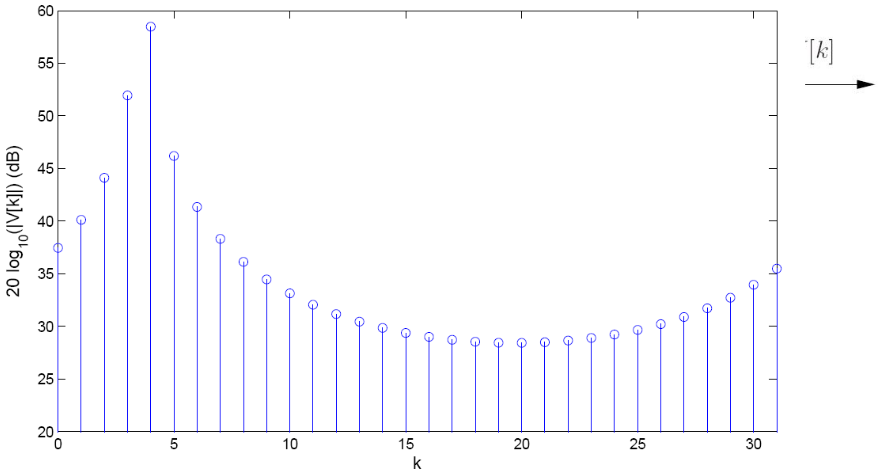

Figure 2: Output $|V[k]|$ in dB.

Problem 2. (25 pts) A continuous-time signal $x_c(t)$ is defined by:

$$x_c(t) = \begin{cases} 1, & \text{if } |t - r \cdot t_0| < t_b, r = 0, \pm 1, \pm 2, \cdots \\ 0, & \text{otherwise.} \end{cases}$$

with $t_b = \frac{t_0}{20}$. $x_1[n]$ is obtained by sampling $x_c(t)$ with a sampling period $T = \frac{t_0}{4}$.

(a) What is $x_1[n]$?

(b) $x_2[n]$ is the finite-length sequence formed by taking $x_1[n]$ for $n = 0, 1, \cdots 7$. What is its 8-point DFT, $X_2[k]$?

(c) Find the DTFT $X_2(e^{j\omega})$ and sketch its magnitude.

(d) Evaluate the Parseval integral $\frac{1}{2\pi} \int_{2\pi} |X_2(e^{j\omega})|^2 d\omega$.

(e) We form $x_3[n] = x_1[n] \circledast h_1[n]$. The DTFT of $h_1[n]$ is $H_1(e^{j\omega}) = 1 - e^{-2j\omega}$. Find the time-domain sequence of $x_3[n]$. Note: $\circledast$ denotes linear convolution.

Solution:

(a) The continuous-time signal is just a pulse of width $0.1t_0$ centered at each integer multiple of t_0. Sampling at every integer multiple of $\frac{t_0}{4}$ yields a periodic sequence $x_1[n] = \{\underline{1}, 0, 0, 0, 1, 0, 0, 0, \cdots\}$.

(b) $x_2[n]$ is simply $\{1, 0, 0, 0, 1, 0, 0, 0\}$ whose DFT can be obtained directly from the DFT equation:

$$\begin{aligned} X_2[k] &= \sum_{n=0}^{N} x_2[n] e^{-j\frac{2\pi kn}{N}} \\ &= 1 + e^{-j\frac{2\pi 4k}{8}} = 1 + e^{-j\pi k} \\ &= 1 + (-1)^k = \{\underline{2}, 0, 2, 0, 2, 0, 2, 0\} \end{aligned}$$

(c) The DTFT is can be obtained if we let ω take the place of $\frac{2\pi k}{N}$:

$$\begin{aligned} X_2(e^{j\omega}) &= \sum_{n=0}^{N} x_2[n] e^{-j\omega n} \\ &= 1 + e^{-j(4\omega)} \\ &= e^{-j(2\omega)} \left(e^{j(2\omega)} + e^{-j(2\omega)} \right) \\ &= 2e^{-j(2\omega)} \cos(2\omega) \end{aligned}$$

(d) Using Parseval relationship, it is easy to figure out that the Parseval integral over $X_2(e^{j\omega})$ equals to the energy of $x_2(n)$, i.e.

$$\frac{1}{2\pi} \int_{2\pi} |X_2(e^{j\omega})|^2 d\omega = \sum_{n=1}^{N} (x_2(n))^2 = 1^2 + 1^2 = 2 \tag{1}$$

(e) We notice that $H_1(e^{j\omega})$ is the DTFT of $\{\underline{1}, 0, -1\}$. Thus, the convolution consists of adding the original sequence to a sign-flipped, delayed-by-two version, or $x_3[n] = \{1, 0, -1, 0, 1, 0, -1, 0, \cdots\}$.

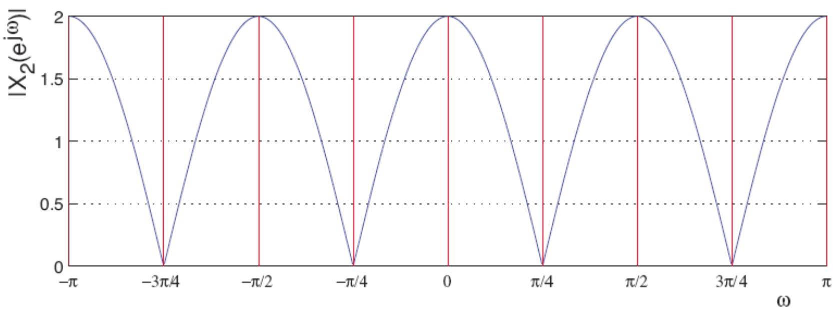

Figure 3: Sketch of $|X_2(e^{j\omega})|$

Problem 3. (25 points) The sampling system is shown in Figure 4(a). With x_t real and with $X(j\omega)$ nonzero only for $\omega_1 < |\omega| < \omega_2$, ω_0 is chosen to be $\frac{1}{2}(\omega_1 + \omega_2)$, and the ideal lowpass filter $H_1(j\omega)$ has cutoff frequency $\frac{1}{2}(\omega_2 - \omega_1)$.

(a) For $X(j\omega)$ as shown in Figure 4(b), sketch $X_p(j\omega)$.

(b) Determine the maximum sampling period T such that $x(t)$ is recoverable from $x_p(t)$.

(c) Determine a system to recover $x(t)$ from $x_p(t)$.

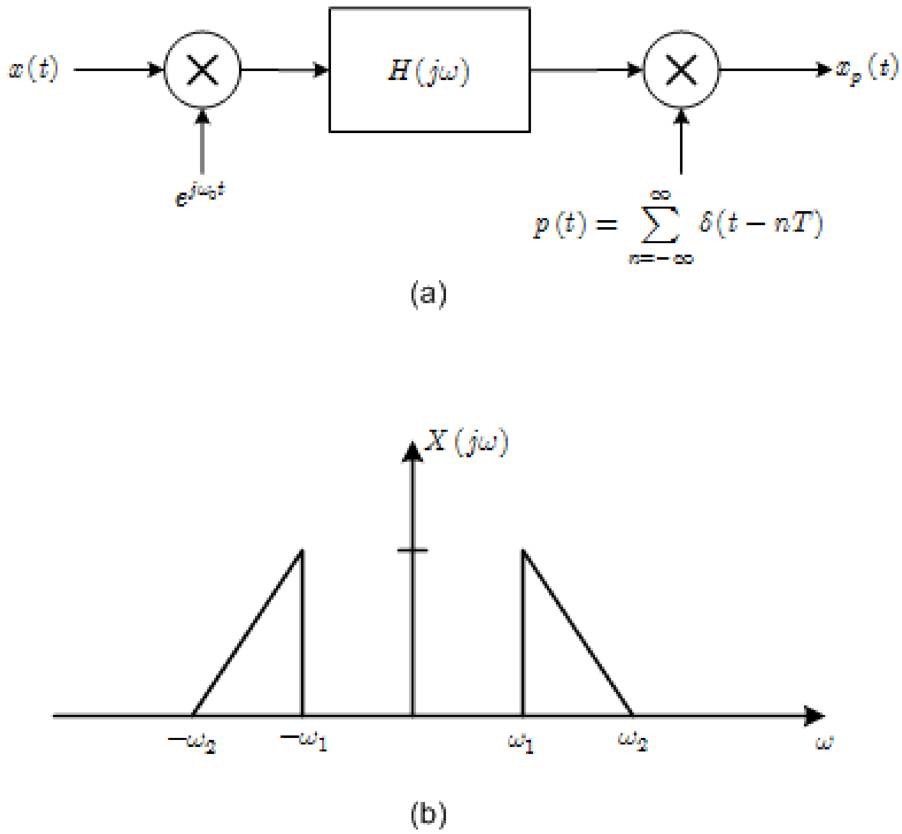

Figure 4: The sampling system and the spectrum of $x(t)$.

Solution:

(a) Let $X_1(j\omega)$ denote the Fourier transform of the signal x_1 obtained by multiplying $x(t)$ with $e^{j\omega_0 t}$. Let $X_2(j\omega)$ be the Fourier transform of the signal $x_2(t)$ obtained at the output of the lowpass filter. Then $X_1(j\omega)$, $X_2(j\omega)$, and $X_p(j\omega)$ are as shown in Figure 5.

(b) The Nyquist rate for the signal $x_2(t)$ is $2 \times (\omega_2 - \omega_1)/2 = \omega_2 - \omega_1$. Therefore, the sampling period T must be at most $\frac{2\pi}{\omega_2 - \omega_1}$ in order to avoid aliasing.

(c) A system that can be used to recover $x(t)$ from $x_p(t)$ is shown in Figure 5.

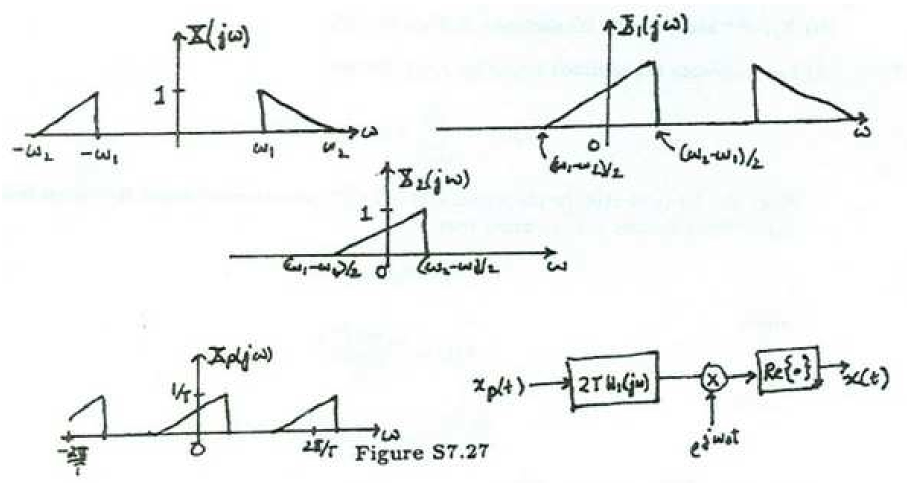

Figure S7.27

Figure 5:

Problem 4. (15 points) The figure below shows the block diagram of several discrete-time LTI systems.

(a) Find the z-transform of systems H_1 and H_2.

(b) Plot the pole-zero diagrams of H_1 and H_2.

(c) Find the impulse responses for systems H_3 and H_4.

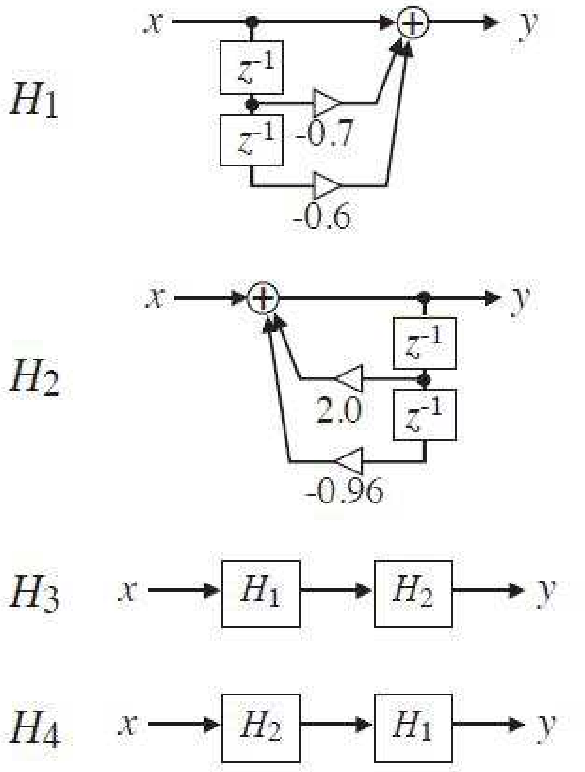

Figure 6: The block diagrams of LTI systems.

Solution:

(a) For system H_1, we have:

$$y(n) = x(n) - 0.7x(n-1) - 0.6x(n-2) \tag{2}$$

$$Y(z) = (1 - 0.7z^{-1} - 0.6z^{-2})X(z) \tag{3}$$

$$H_1(z) = \frac{Y(z)}{X(z)} = (1 - 1.2z^{-1})(1 + 0.5z^{-1}) \tag{4}$$

i.e. a system with zeros at $z = -0.5, 1.2$. Since it has no poles, the ROC is the entire z-plane. For H_2, we have:

$$y(n) = x(n) + 2.0y(n-1) - 0.96y(n-2) \tag{5}$$

$$Y(z) = X(z) + 2.0z^{-1}Y(z) - 0.96z^{-2}Y(z) \tag{6}$$

Therefore,

$$Y(z)(1 - 2.0z^{-1} + 0.96z^{-2}) = X(z) \tag{7}$$

and

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{(1 - 1.2z^{-1})(1 - 0.8z^{-1})} \tag{8}$$

i.e. a system with poles at $z = 0.8, 1.2$. Since it is a causal system, the ROC is the exterior of the largest pole radius, i.e. $|z| > 1.2$.

(b) Please refer to Figure 7.

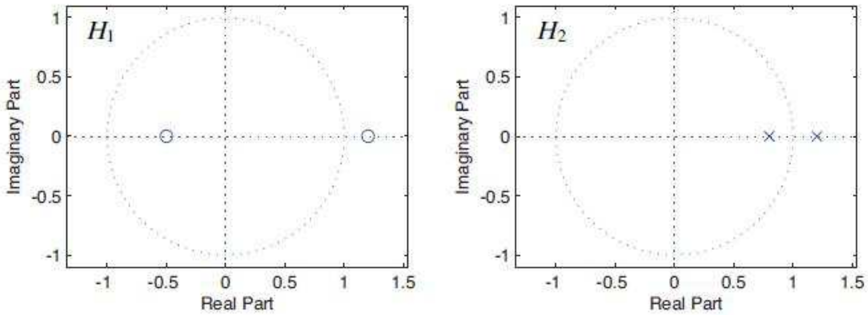

Figure 7: The zeros and poles.

(c) Both systems H_3 and H_4 consist of the sequential application of H_1 and H_2, the only difference being the order in which they are applied. According to what we have looked at in class, convolution is commutative, so it should make no difference in what order they are applied. In both cases, the system function is the product of the two component systems i.e.

$$\begin{aligned} H_3(z) &= H_1(z)H_2(z) && (9)\\ &= (1 - 1.2z^{-1})(1 + 0.5z^{-1})\frac{1}{(1 - 1.2z^{-1})(1 - 0.8z^{-1})} && (10)\\ &= \frac{1 + 0.5z^{-1}}{1 - 0.8z^{-1}} && (11) \end{aligned}$$

where we cancel the common root at $z = 1.2$ in numberator and denominator ("pole-zero cancellation"). To find the impulse response, we can simply find the inverse z-transform of this system function, i.e.

$$h_3(n) = h_a(n) + 0.5h_a(n - 1) \tag{12}$$

where

$$h_a(n) = Z^{-1}(\frac{1}{1 - 0.8z^{-1}}) = 0.8^n u(n) \tag{13}$$

so

$$h_3(n) = 0.8^n u(n) + 0.5 \times 0.8^{n-1} u(n - 1) \tag{14}$$

Problem 5. (20 points) Consider the block diagram in the figure below. The input $x(t)$ is sampled at rate of F_s =12 KHz. The Fourier transform of $x(t)$ and the transfer function $H(e^{j\omega})$ are plotted in Figure 8.

(a) Plot $Y(e^{j\omega})$.

(b) Plot $Z(e^{j\omega})$.

(c) Find the energy of $z(n)$.

(d) Find $z(n)$.

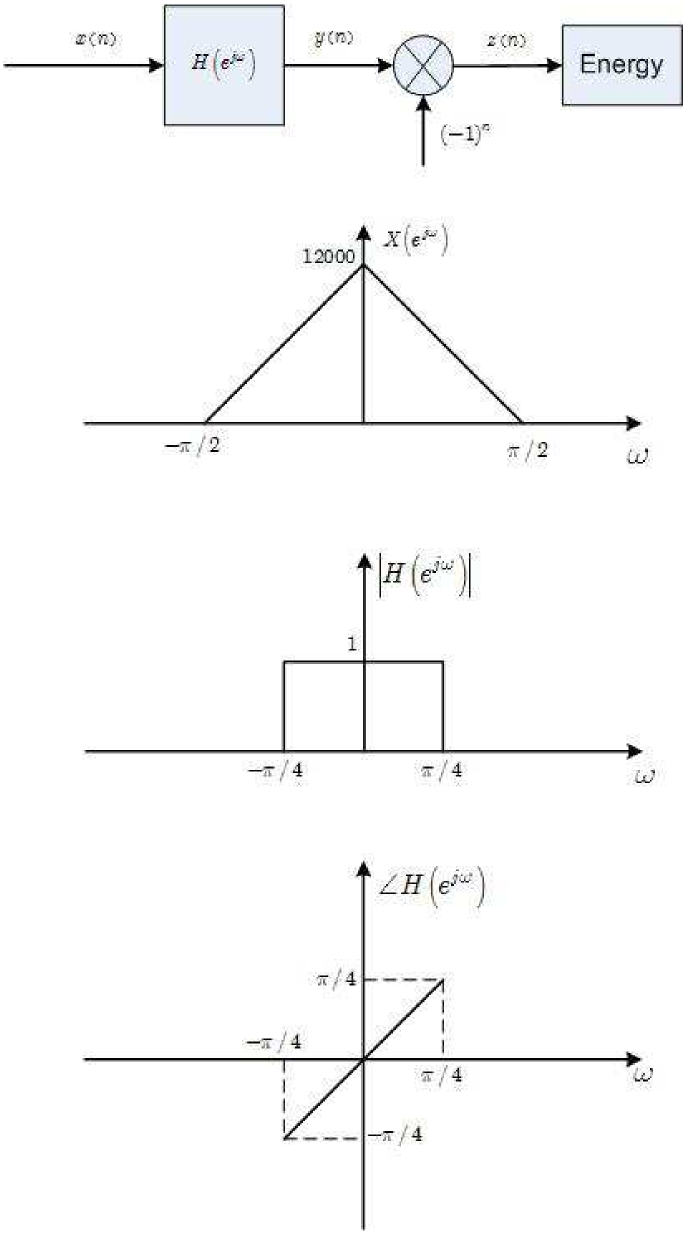

Figure 8: The system diagram and the spectra of $x(n)$ and $h(n)$.

Solution:

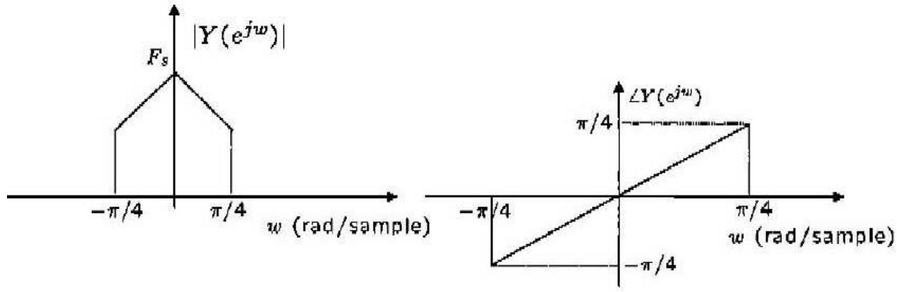

Figure 9:

(a)

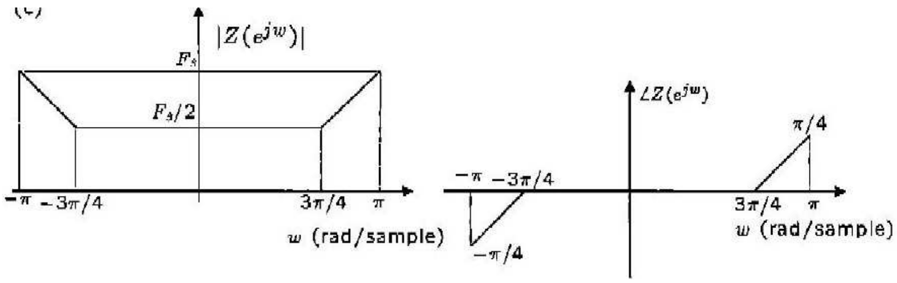

Figure 10:

(b)

(c)

$$
\begin{aligned}
E &= \sum_{n=-\infty}^{\infty} |z(n)|^2 = \frac{1}{2\pi}\int_{2\pi} |Z(e^{j\omega})|^2 d\omega && (15)\\
&= \frac{2}{2\pi}\int_0^{\pi/4} (-\frac{2\times 12000}{\pi}\omega + 12000)^2 d\omega && (16)\\
&= \frac{1}{\pi}[\frac{4\times 12000^2}{3\pi^2}\omega^3 + 12000^2 - \frac{2\times 12000^2}{\pi}\omega^2]_0^{\pi/4} && (17)\\
&= \frac{7}{48}12000^2 = 21\times 10^6. && (18)
\end{aligned}
$$

(d) Let us first find $y(n)$.

$$
\begin{aligned}
y(n) &= \frac{1}{2\pi}\int_{2\pi} Y(e^{j\omega})e^{j\omega n}d\omega && (19)\\
&= \frac{1}{2\pi}\int_0^{\pi/4}(-\frac{24000}{\pi}\omega + 12000)e^{j\omega}e^{j\omega n}d\omega && (20)\\
&\quad +\frac{1}{2\pi}\int_{-\pi/4}^{0}(\frac{24000}{\pi}\omega + 12000)e^{j\omega}e^{j\omega n}d\omega && (21)\\
&= -\frac{12000}{\pi^2}\int_0^{\pi/4}\omega(e^{j\omega(n+1)} + e^{-j\omega(n+1)})d\omega && (22)\\
&\quad +\frac{12000}{2\pi}\int_0^{\pi/4}(e^{j\omega(n+1)} + e^{-j\omega(n+1)})d\omega && (23)\\
&= -\frac{24000}{\pi^2}\int_0^{\pi/4}\omega cos((n+1)\omega)d\omega + \frac{12000}{\pi}\int_0^{\pi/4}cos((n+1)\omega)d\omega && (24)\\
&= -\frac{24000}{\pi^2(1+n)^2}[cos((n+1)\frac{\pi}{4}) + (1+n)\frac{\pi}{4}sin((n+1)\frac{\pi}{4}) - 1] && (25)\\
&\quad +\frac{12000}{\pi(1+n)}[sin((n+1)\frac{\pi}{4})]. && (26)
\end{aligned}
$$

and now $z(n)$ is given by

$$z(n) = (-1)^n y(n) \tag{27}$$