EE113: Digital Signal Processing Prof. Mihaela van der Schaar
Final Exam (Version A) Spring 2009

Name: **Student ID:**

Please read the following instructions:

(i) Fill in your name and student ID above.

(ii) There are 5 problems on the exam. The points for each problem are specified next to the problem statement.

(iii) Space is provided for you to solve each problem. If needed, extra space is available at the end of the exam booklet.

(iv) You are allowed to use two sheets of notes (front and back).

(v) No calculators and no electronic devices are allowed.

(vi) Please do not remove pages from the exam booklet.

(vii) Please keep your eyes on your own exam, and do not talk with your neighbors.

(viii) Please clearly box your answers.

(ix) **Good luck!**

1. (15 pts) The system T in Figure 1 is known to be *time-invariant*. When the inputs to the system are $x_1(n)$, $x_2(n)$, and $x_3(n)$, the responses of the system are $y_1(n)$, $y_2(n)$ and $y_3(n)$, respectively, as shown. (For each sequence, the samples that are not shown are all 0.)

   (a) (5 pts) Determine whether the system T could be linear. Briefly justify your answer.

   (b) (5 pts) If the input $x(n)$ to the system is $\delta(n)$, what is the system response $y(n)$?

   (c) (5 pts) What are all possible inputs $x(n)$ for which the response of the system T can be determined from the given information above?

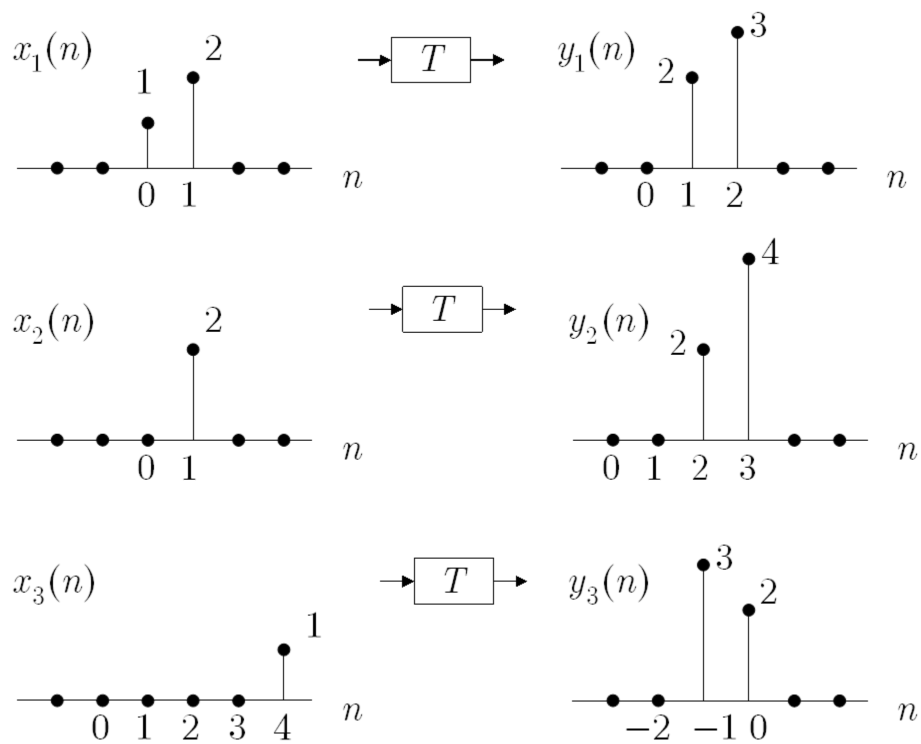

Figure 1: Input-output pairs of the system T

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2. (25 pts) For the block diagram shown in Figure 2:

$$x(n) = 16\left(\frac{\sin\frac{\pi}{8}n}{\pi n}\right)^2 \cos\frac{\pi}{4}n + \frac{\sin\frac{\pi}{4}n}{\pi n}$$

$$H(e^{j\omega}) = \begin{cases} \frac{1}{\frac{4}{\pi}|\omega|+1} & |\omega| \le \frac{\pi}{4} \\ 0 & \frac{\pi}{4} < |\omega| \le \pi. \end{cases}$$

(a) (10 pts) Plot the DTFTs of the signals at points A, B, and C of Figure 2.

(b) (5 pts) Find an expression for $v(n)$.

(c) (5 pts) Evaluate the energy of the output sequence $y(n)$.

(d) (5 pts) Find the 8-point DFT of $x(n)$ and $y(n)$.

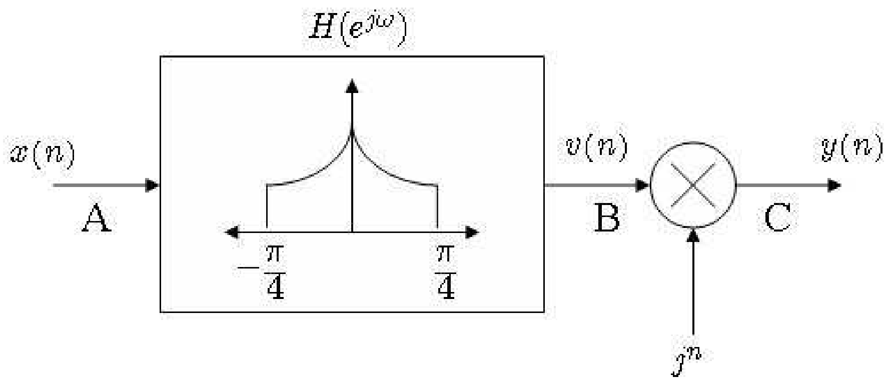

Figure 2: Block diagram

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3. (25 pts) Consider the continuous time signal

$$x(t) = \sin(500\pi t)\sin(1000\pi t)$$

(a) (5 pts) Find the minimum frequency required for sampling $x(t)$ in order to avoid aliasing.

(b) (5 pts) If $x(t)$ is sampled at the rate of 750 Hz, find the discrete time sequence $x(n)$ that corresponds to the sampled signal.

(c) (5 pts) Identify the aliased frequency components in $x(n)$.

(d) (5 pts) The discrete-time sequence $x(n)$ obtained in part (b) is now processed by the following relaxed LTI system:

$$y(n) = \frac{1}{2}y(n-1) + x(n).$$

Find $y(n)$.

(e) (5 pts) For what sampling frequency is the discrete time signal $x(n)$ always zero?

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Please clearly box your answers.

4. (20 pts) $x(n)$ is a sequence with a finite number of nonzero values all of which appear in Figure 3. Its DTFT, $X(e^{j\omega})$, is formed, and the real part is inverted to make a new sequence $y(n) = \text{IDTFT}(\text{Real}\{X(e^{j\omega})\})$.

   (a) (5 pts) What is $y(n)$?

   (b) (5 pts) What is its DTFT, $Y(e^{j\omega})$?

   (c) (5 pts) Express the DTFT of the difference $x(n) - y(n)$ in terms of $X(e^{j\omega})$.

   (d) (5 pts) If $Y(e^{j\omega}) = (G(e^{j\omega}))^2$, what is $g(n)$, the IDTFT of $G(e^{j\omega})$.

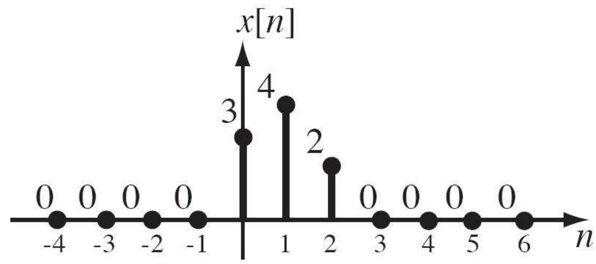

Figure 3: Sequence $x(n)$

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5. (25 pts) Consider the cascade of LTI systems shown in Figure 4, where

$$\begin{aligned} h_1(n) &= (-1)^n u(n) \\ h_2(n) &= u(n) - u(n-4) \\ h_3(n) &= \delta(n) - \delta(n-2) \end{aligned}$$

(a) (10 pts) Find the impulse response of the system in Figure 4.

(b) (5 pts) Compute the DTFT of $h(n)$, $H(e^{j\omega})$.

(c) (5 pts) Plot the magnitude and phase of $H(e^{j\omega})$.

(d) (5 pts) Determine the output $y(n)$ of the overall system if the input is $x(n) = \cos(\frac{\pi}{2}n)$.

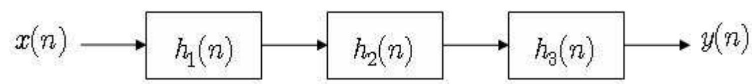

Figure 4: Cascade of LTI systems

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Solutions:

1. (a) We first observe that $x_1(n) = x_2(n) + x_3(n+4)$. Because the system T is time-invariant, the output of the system is $y_3(n+4)$ when the input is $x_3(n+4)$. If the system were linear, then we should have $y_1(n) = y_2(n) + y_3(n+4)$. This is clearly not true, and therefore the system **is not linear**.

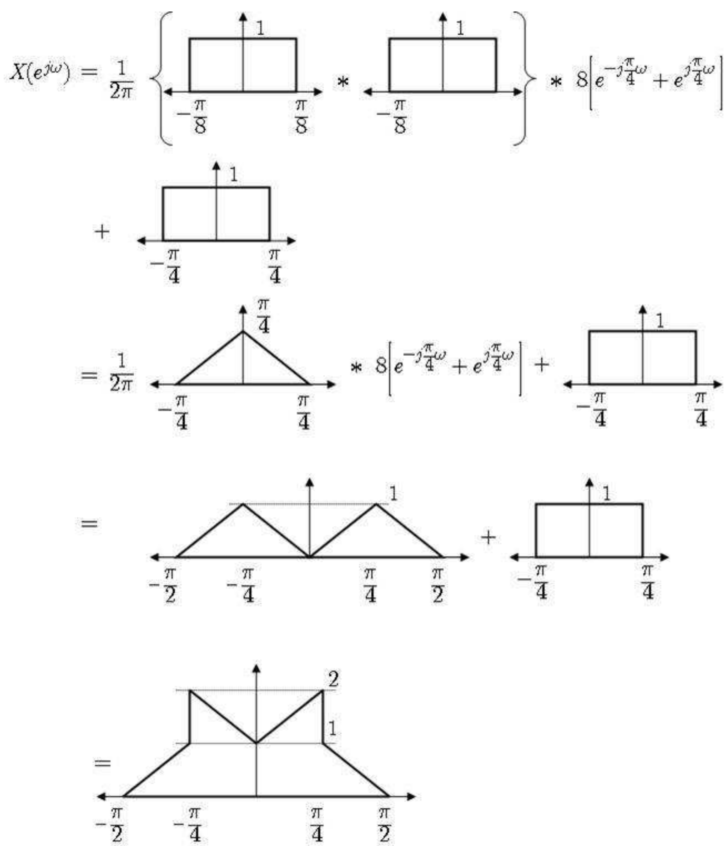
   (b) If $x(n) = \delta(n) = x_3(n+4)$, then $y(n) = y_3(n+4)$ because the system T is time-invariant.

   (c) We can evaluate the response of the system T to $x_1(n-k)$, $x_2(n-k)$, and $x_3(n-k)$ for all integers k

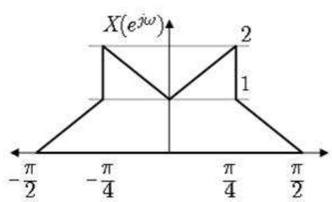
2. (a) Since

$$x(n) = 16\left(\frac{\sin\frac{\pi}{8}n}{\pi n}\right)^2 \cos\frac{\pi}{4}n + \frac{\sin\frac{\pi}{4}n}{\pi n}$$

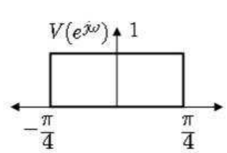
The DTFTs of the signals at points A, B, and C (i.e. $X(e^{j\omega})$, $V(e^{j\omega})$, and $Y(e^{j\omega})$, respectively) can be evaluated as illustrated below.



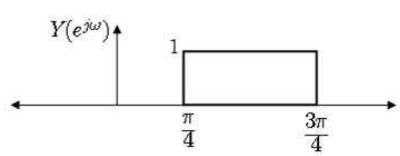
A:



B:



C:



(b) $v(n) = \frac{\sin \frac{\pi}{4} n}{\pi n}$.

(c) Using Parseval's relation

$$\sum_{n=-\infty}^{\infty} |y(n)|^2 = \frac{1}{2\pi} \int_{2\pi} |Y(e^{j\omega})|^2 d\omega = \frac{1}{2\pi} \frac{\pi}{2} = \frac{1}{4}$$

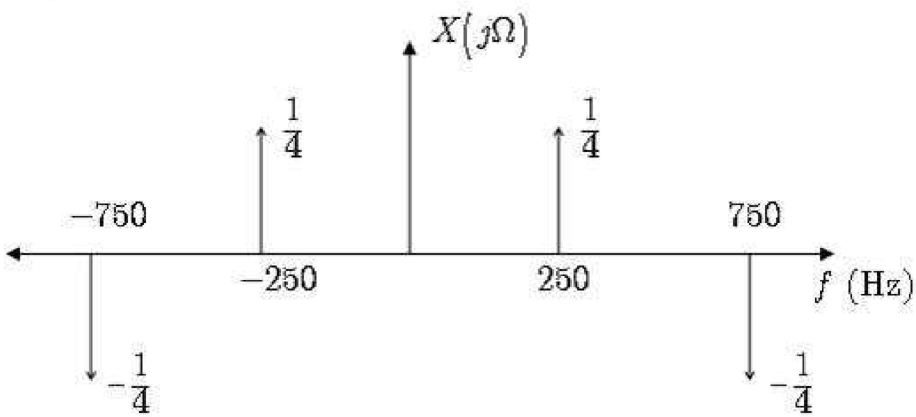
(d) The 8-point DFT of $Y(k) = Y(e^{j\omega})|_{\omega = \frac{2\pi}{8} k}$ for $k = 0, \cdots, 7$ is:

$$Y(0) = Y(4) = Y(5) = Y(6) = Y(7) = 0$$
$$Y(1) = Y(2) = Y(3) = 1$$

The 8-point DFT of $X(k) = X(e^{j\omega})|_{\omega = \frac{2\pi}{8} k}$ for $k = 0, \cdots, 7$ is:

$$X(0) = 1$$
$$X(1) = X(7) = 2$$
$$X(2) = X(3) = X(4) = X(5) = X(6) = 0$$

3. We have $x(t) = \sin(500\pi t)\sin(1000\pi t) = \frac{1}{2}\cos(500\pi t) - \frac{1}{2}\cos(1500\pi t)$. $X(j\Omega)$, the CTFT of $x(t)$, is displayed below.



(a) The Nyquist rate is $F_s = 2 * 750 = 1500$ Hz.

(b) We have $T_s = \frac{1}{750}$ and $x(n) = x(t)|_{t=nT_s} = \frac{1}{2}\cos(\frac{2\pi}{3}n) - \frac{1}{2}\cos(2\pi n) = \frac{1}{2}\cos(\frac{2\pi}{3}n) - \frac{1}{2}$

(c) $f = 750$ Hz or $\omega = 2\pi$ since it is greater than half of the sampling frequency or greater than π.

(d) In the DTFT domain:

$$\begin{aligned} Y(e^{j\omega}) &= \frac{1}{2}e^{-j\omega}Y(e^{j\omega}) + X(e^{j\omega}) \\ H(e^{j\omega}) &= \frac{1}{1-\frac{1}{2}e^{-j\omega}} \\ X(e^{j\omega}) &= \frac{\pi}{2}[\delta(\omega - \frac{2\pi}{3}) + \delta(\omega + \frac{2\pi}{3})] - \pi \end{aligned}$$

We therefore have:

$$\begin{aligned} Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) \\ &= \frac{\frac{\pi}{2}[\delta(\omega - \frac{2\pi}{3}) + \delta(\omega + \frac{2\pi}{3})] - \pi}{1-\frac{1}{2}e^{-j\omega}} \\ &= \frac{\pi}{2}\left[\frac{\delta(\omega - \frac{2\pi}{3})}{1-\frac{1}{2}e^{-j\frac{2\pi}{3}}} + \frac{\delta(\omega + \frac{2\pi}{3})}{1-\frac{1}{2}e^{j\frac{2\pi}{3}}}\right] - \frac{\pi}{1-\frac{1}{2}e^{-j\omega}} \end{aligned}$$

and,

$$y(n) = \frac{e^{j\frac{2\pi}{3}n}}{4-2e^{-j\frac{2\pi}{3}}} + \frac{e^{-j\frac{2\pi}{3}n}}{4-2e^{j\frac{2\pi}{3}}} - \pi(\frac{1}{2})^n u(n)$$

(e) There are mutliple solutions to this problem (e.g. $T_s = \frac{k}{1000}$, for $k = 0, 1, 2\cdots$). For $T_s = \frac{1}{500}$ we have

$$x(n) = x(t)|_{t=nT_s} = \frac{1}{2}\cos(\pi n) - \frac{1}{2}\cos(3\pi n) = 0$$

4. We have

$x[n] = \{\underline{3}, 4, 2\}$ (underline indicates $n = 0$ point).

(a) $Y(e^{j\omega}) = \mathbb{R}\mathrm{e}\{X(e^{j\omega})\}$, and a real Fourier transform corresponds to a conjugate-symmetric sequence. We can express $x[n]$ as the sum of a conjugate-symmetric part $x_{cs}[n] = 1/2(x[n] + x^*[-n])$ and a conjugate antisymmetric part $x_{ca}[n] = 1/2(x[n] - x^*[n])$, and thus $y[n] = x_{cs}[n] = \{1, 2, \underline{3}, 2, 1\}$.

(b) The DTFT,

$$Y(e^{j\omega}) = \sum_{\forall n} x[n]e^{j\omega n} = 3 + 2(e^{j\omega} + e^{-j\omega}) + (e^{2j\omega} + e^{-2j\omega})$$
$$= 3 + 4cos\omega + 2cos2\omega$$

We note that it is pure real, as expected.

(c) Since $Y(e^{j\omega})$ is the real part of $X(e^{j\omega})$, $DTFT(x[n]-y[n]) = X(e^{j\omega})-Y(e^{j\omega}) = \mathbb{I}\mathrm{m}\{X(e^{j\omega})\}$, the imaginary part of the original sequence (corresponding to the conjugate-antisymmetric, or in this case odd, part of the original sequence).

(d) The key to this part is to notice that $y[n]$ is a triangular pulse, which we have seen before as the convolution of two rectangular pulses. Thus, we expect there to be a solution with $g[n]$ as a three-point rectangular pulse. However, let's do the math:

$$\begin{aligned} G(e^{j\omega})^2 = Y(e^{j\omega}) &= 3 + 4cos\omega + 2cos2\omega \\ &= 3 + 4cos\omega + 2(2cos^2\omega - 1) \\ &= 1 + 4cos\omega + 4cos^2\omega \\ &= (\pm(1 + 2cos\omega))^2 \end{aligned}$$

Thus,

$$G(e^{j\omega}) = \pm(1 + 2cos\omega) = \pm(1 + e^{j\omega} + e^{-j\omega})$$
$$\Rightarrow g[n] = \pm(\delta[n-1] + \delta[n] + \delta[n+1])$$

CORRECTION: In part (c) above, $X(e^{j\omega}) - Y(e^{j\omega}) = Im\left[X(e^{j\omega})\right]$ should be written as $X(e^{j\omega}) - Y(e^{j\omega}) = jIm\left[X(e^{j\omega})\right]$

5. By definition, the impulse response is the response of a system when the input is set to an impulse.

   (a) Because we have a cascade of LTI systems, we know that

   $$h(n) = \delta(n) * h_1(n) * h_2(n) * h_3(n)$$

   We know that convolving anything with $\delta(n)$ gives you the same thing, therefore

   $$\begin{aligned} h(n) &= h_1(n) * h_2(n) * h_3(n) \\ &= h_1(n) * h_3(n) * h_2(n) \end{aligned}$$

   because convolution is commutative. We first convolve $h_1(n)$ and $h_3(n)$:

   $$\begin{aligned} h_1(n) * h_3(n) &= (-1)^n u(n) * [\delta(n) - \delta(n-2)] \\ &= (-1)^n u(n) * \delta(n) - (-1)^n u(n) \delta(n-2) \\ &= (-1)^n u(n) - (-1)^{n-2} u(n-2) \\ &= (-1)^n [u(n) - u(n-2)] \\ &= (-1)^n [\delta(n) + \delta(n-1)] \\ &= \delta(n) - \delta(n-1) \end{aligned}$$

   We now convolve this result with $h_2(n)$:

   $$\begin{aligned} h(n) &= [\delta(n) - \delta(n-1)] * [\delta(n) + \delta(n-1) + \delta(n-2) + \delta(n-3)] \\ &= \delta(n) - \delta(n-4) \end{aligned}$$

   (b) The frequency response of $h(n) = \delta(n) - \delta(n-4)$ is:

   $$\begin{aligned} H(e^{j\omega}) &= 1 - e^{-j\omega 4} \\ &= e^{-j\omega 2}[e^{j\omega 2} - e^{-j\omega 2}] \\ &= 2je^{-j\omega 2}\sin(2\omega), \text{ but } j = e^{j\frac{\pi}{2}} \\ &= e^{-j2\omega} e^{j\frac{\pi}{2}} 2\sin(2\omega) \\ &= e^{j(-2\omega + \frac{\pi}{2})} 2\sin(2\omega) \end{aligned}$$

   (c) The magnitude response $|H(e^{j\omega})|$ is:

   $$\begin{aligned} |H(e^{j\omega})| &= |e^{j(-2\omega + \frac{\pi}{2})} 2\sin(2\omega)| \\ &= |2\sin(2\omega)| \end{aligned}$$

   and the phase response is

   $$\angle H(e^{j\omega}) = -2\omega + \frac{\pi}{2}$$

   (d) We know the response to a sinusoid of a real LTI system That is, if $x(n) = A\cos(\omega_0 n)$, then

   $$y(n) = A|H(e^{j\omega})|\cos[\omega_0 n + \angle H(e^{j\omega})].$$

   But, we have $x(n) = 2\cos(\frac{\pi}{2}n)$, meaning our $\omega_0 = \frac{\pi}{2}$. At $\frac{\pi}{2}$, $|H(e^{j\omega})|_{\frac{\pi}{2}}$=0. Thus,

   $$y(n) = 0.$$