ECE113, Digital Signal Processing
UCLA Spring 2018
Midterm Exam
05/07/2018
Time Limit: 2 hours

Name: Solutions

(a) This exam booklet contains 16 pages (including this cover page) and 6 problems. Total of points is 100.

(b) ONE textbook of your choice (hard copy only) can be open. ALL notes must remain closed. You are also allowed to have a single double-sided letter-sized cheat sheet, if needed.

(c) Simple calculators are allowed. But no fancy calculators, smartphones, or any other smart devices.

(d) Please fully justify your answers and clearly show ALL the intermediate steps in all your solutions. And, when appropriate, box your final answer.

(e) Please do NOT write your answers on the back of any pages. Answers written on the back will NOT be graded.

(f) **Good Luck...**

Grade Table (for instructor use only)

Question	Points	Score
1	10	
2	17	
3	12	
4	18	
5	25	
6	18	
Total:	100	

1. (10 points) **Quick Review**
Carefully read each statement below and identify it as *True* or *False* by clearly writing **T** or **F** in the box.

- F ☐ Discrete-time sinusoids are always periodic in time.
- T ☐ Discrete-time Fourier Transforms are always periodic in frequency.
- F ☐ A discrete-time accumulator is BIBO stable.
- F ☐ Zero-padding a discrete-time sequence changes both the *shape* and the *resolution* of its frequency spectrum by increasing the number of points, N, for its N-point DFT.
- T ☐ Any discrete-time sequence of length N or less can always be represented by its N-point DFT.
- T ☐ Downsampling a lowpass sequence by a factor D will always lead to aliasing if the discrete-time sequence has any frequency components within $[\frac{\pi}{D}, \pi]$ range.
- F ☐ Upsampling by a factor I will lead to aliasing if the discrete-time sequence has any frequency components within $[\frac{\pi}{I}, \pi]$ range.
- F ☐ An N-point *circular convolution* of a sequence with length N_1 with another sequence of length N_2 will always be equal to the *linear convolution* of the two sequences within the range $[0, N-1]$ as long as $N \geq \max(N_1, N_2)$.
- T ☐ FIR filters are always BIBO stable.
- F ☐ The *minimum* sampling rate to avoid aliasing for a real-valued *bandpass* signal with its single-side band (i.e., over positive frequencies) limited to B Hz would always be $2B$.

2. (17 points) **LCCDE, Direct Form Structures**:

A second-order LTI system is described by the following Linear Constant Coefficient Difference Equation:

$$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1) \quad (1)$$

Assume the system is at rest, i.e., $y(-2) = y(-1) = 0$.

(a) (4 points) Write the characteristic equation of the system, find the natural frequencies (modes), and write the form of the homogeneous response of the system.

Charac. Eq.: $\lambda^2 - 3\lambda - 4 = (\lambda+1)(\lambda-4) = 0$

$\Longrightarrow \lambda_1 = -1, \; \lambda_2 = 4$: Modes (natural frequencies)

$\Longrightarrow y_H(n) = C_1(-1)^n + C_2(4)^n$: Homogenous Response

(b) (6 points) Find the complete system response, $y(n), n \geq 0$, to the input sequence $x(n) = 2^n u(n)$, where $u(n)$ is the unit step sequence.

$y_p(n) = K.2^n u(n) \Longrightarrow K2^n u(n) - 3K2^{n-1}u(n-1) - 4K2^{n-2}u(n-2)$

$= 2^n u(n) + 2 \times 2^{n-1} u(n-1)$

For $n \geq 2$: $K2^n - 3K2^{n-1} - 4K2^{n-2} = 2^n + 2^n$

$\Longrightarrow \cancel{2^n}\left(K - \frac{3K}{2} - \frac{4K}{4}\right) = \cancel{2^n} \times 2 \Longrightarrow K = -\frac{4}{3}$

$\Longrightarrow y_p(n) = \left(-\frac{4}{3}\right)2^n u(n)$

$\Longrightarrow y(n) = y_p(n) + y_H(n) = -\frac{4}{3}2^n + C_1(-1)^n + C_2(4)^n, \; n \geq 2$

$y(-1) = y(-2) = 0 \Longrightarrow y(0) = x(0) = 1$

$y(1) - 3y(0) = x(1) + 2x(0) \Longrightarrow y(1) = 3 + 2 + 2 = 7$

$\Longrightarrow \begin{cases} -\frac{4}{3} + C_1 + C_2 = 1 \\ -\frac{8}{3} - C_1 + 4C_2 = 7 \end{cases} \Longrightarrow \begin{cases} C_1 = -\frac{1}{15} \\ C_2 = \frac{12}{5} \end{cases}$

$\Longrightarrow y(n) = -\frac{4}{3} \cdot 2^n - \frac{1}{15}(-1)^n + \frac{12}{5} \cdot 4^n \quad , \; n \geq 0$

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(c) (2 points) How would the form of the response change if the input was $x(n) = 4^n u(n)$ instead? You don't need to find the values of the constants again. Just indicate if you think the form of the response would change and if so, how?

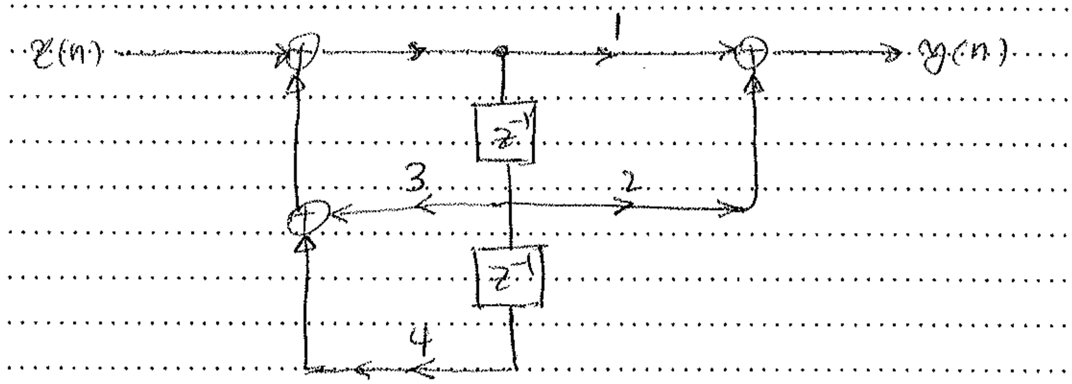
Since 4 is one of the modes, an exponential input of the form $4^n u(n)$ would effectively make the mode 4 be excited to the 2nd order, resulting in a $Kn4^n u(n)$ term in the particular response (see Example 2.4.9 in R1)

(d) (5 points) Determine and draw the signal flow graph for the Direct Form II realization of this system.

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

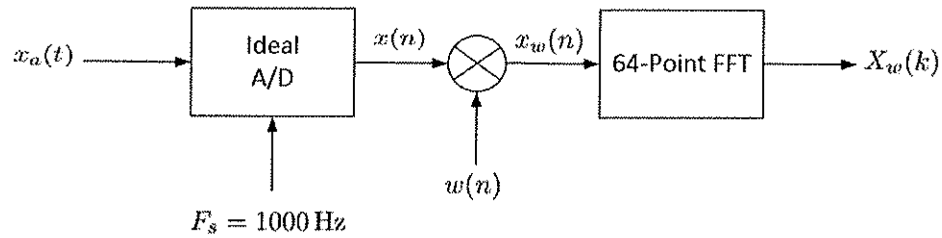
In this case, we have: $a_1 = -3$, $a_2 = -4$, $b_0 = 1$, $b_1 = 2$

Direct Form II:



3. (12 points) **Spectral Analysis**:

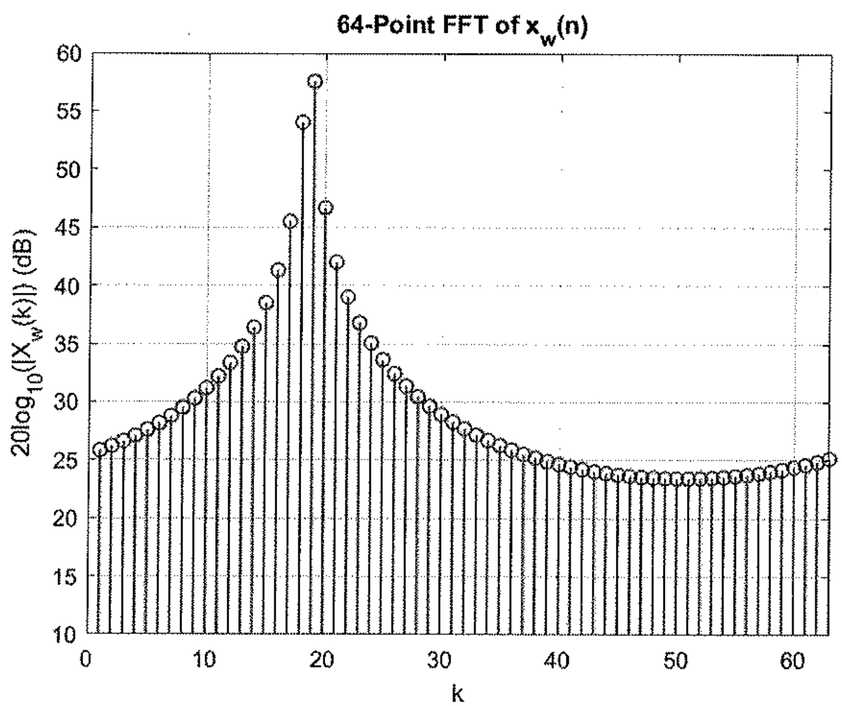
A system for discrete-time spectral analysis of a continuous-time signal is shown below:



where $w(n)$ is a rectangular window:

$$w(n) = \begin{cases} \frac{1}{64}, & \text{for } 0 \le n \le 63 \\ 0, & \text{otherwise} \end{cases}$$

We have obtained the 64-point FFT, $X_w(k)$, the magnitude of which is shown below with the vertical axis in dB scale:



The associated continuous-time input signal, $x_a(t)$, could be one or more of the following signals. Identify which one(s) of the signals below could have produced this FFT. And clearly explain your reasoning for your choice(s).

Hint: Do not try to analyze the signals one at a time. Instead, first look at all the signals and, from what you know about DFT's, try to divide and conquer!

- [x] $x_{a1}(t) = 10\cos(550\pi t)$
- [x] $x_{a2}(t) = 1000\cos(550\pi t)$
- [x] $x_{a3}(t) = 10e^{j550\pi t}$
- [x] $x_{a4}(t) = 1000e^{j550\pi t}$
- [x] $x_{a5}(t) = 10\cos(531.25\pi t)$
- [x] $x_{a6}(t) = 1000\cos(531.25\pi t)$
- [x] $x_{a7}(t) = 1000e^{j531.25\pi t}$
- [x] $x_{a8}(t) = 1000e^{j562.5\pi t}$
- [x] $x_{a9}(t) = 1000\cos(562.5\pi t)$
- [x] $x_{a10}(t) = 1000e^{j2562.5\pi t}$
- [x] $x_{a11}(t) = 1000\cos(2550\pi t)$
- [x] $x_{a12}(t) = 1000e^{j2550\pi t}$

* Only one peak. So input cannot include cos function. That excludes $x_{a_1}, x_{a_2}, x_{a_5}, x_{a_6}, x_{a_9}, x_{a_{11}}$

* The amplitude of the rectangular window is set to $\frac{1}{N} = \frac{1}{64}$. So it normalized the DFT magnitudes. Yet the peak is near 60dB = 1000. So the input signal cannot have an amplitude of 10. That excludes x_{a_3} (in addition to x_{a_1} and x_{a_5} which had already been excluded).

* The FFT does not show a single nonzero sample. So we clearly have leakage. As such the input freq. cannot coincide with any of the freq bins which are located at multiples of $\frac{F_s}{N} = \frac{1000}{64} = 15.625$ Hz

That excludes x_{a_7} ($f_0 = \frac{531.25}{2} = 17 \times 15.625$) and x_{a_8} ($f_0 = \frac{562.5}{2} = 18 \times 15.625$) ~~and~~ Also $x_{a_{10}}$ ($\frac{2562.5}{2} = 82 \times 15.625$)

* would alias back onto one of the bins and could not have been our input signal.

* That leaves us with $\boxed{x_{a_4}}$ and $\boxed{x_{a_{12}}}$ both of which could have been our input. Notice that the freq. of $x_{a_{12}}$ is exactly $F_s = 1000$ Hz apart from the freq of x_{a_4} and as such would alias back to exactly the same samples in the freq domain.

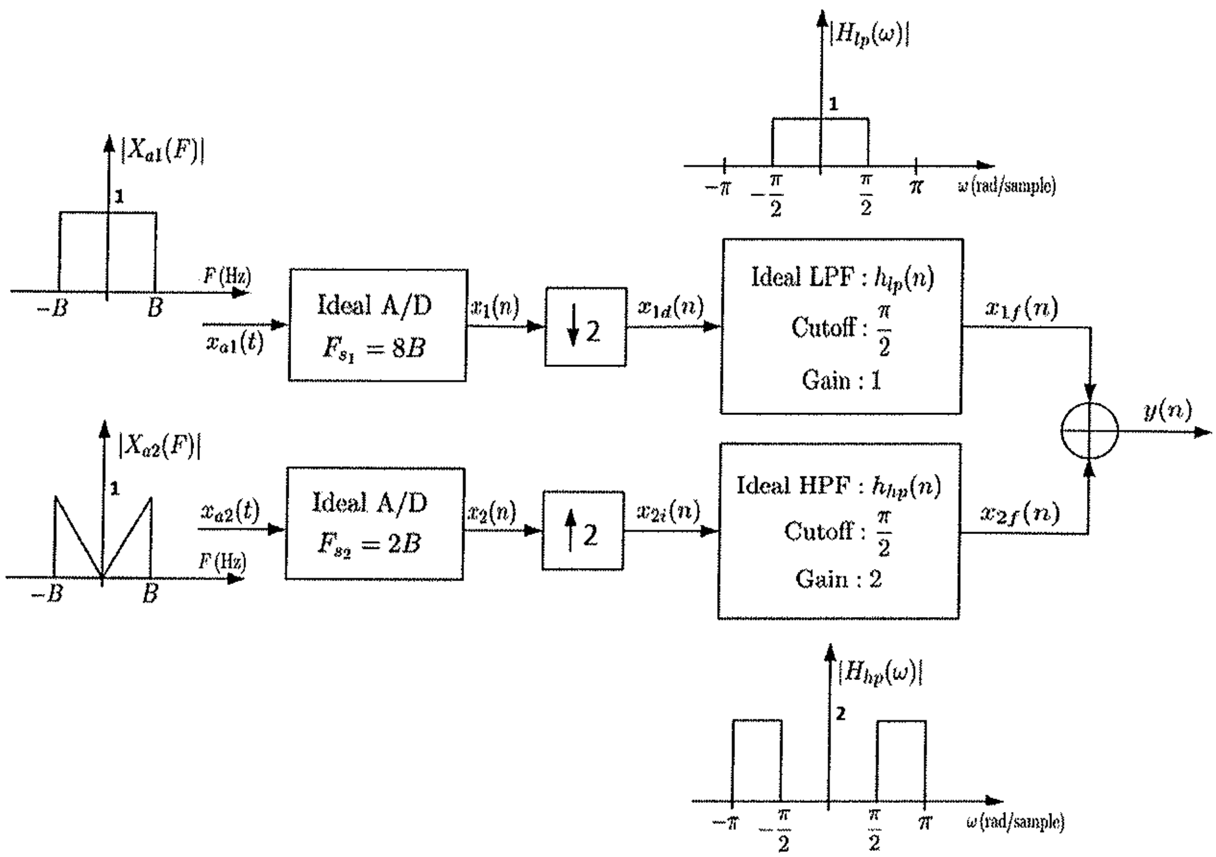
4. (18 points) **DTFT, Sample Rate Conversion:**

Consider the system below where two continuous-time signals $x_{a1}(t)$ and $x_{a2}(t)$, with the given spectrum (CTFT) are first sampled at different rates, then downsampled and upsampled respectively in order to have the same sampling rate, then passed through an ideal lowpass and an ideal highpass filter respectively, and finally added up, forming a composite signal $y(n)$.

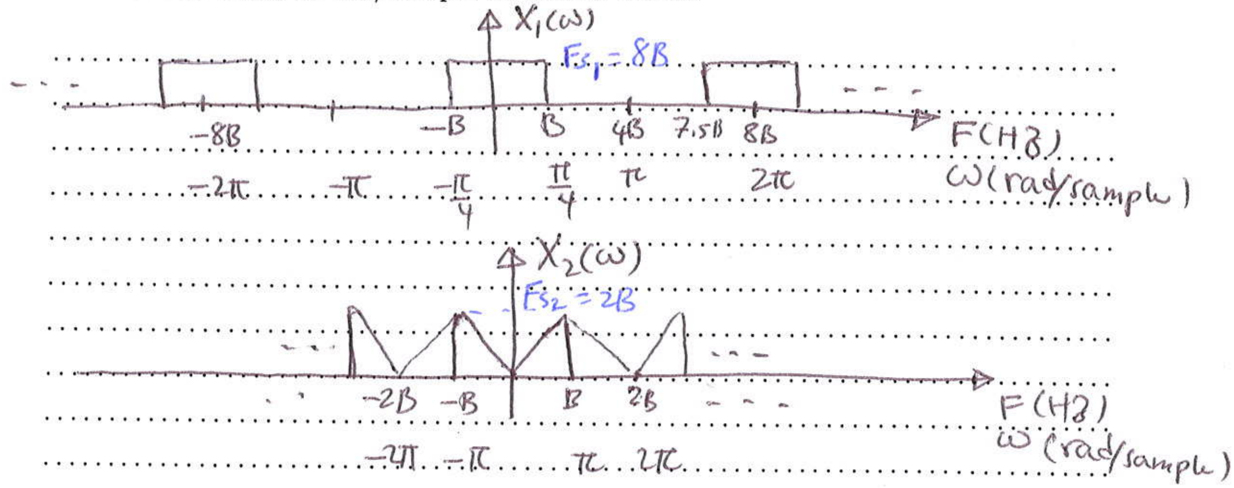
The impulse responses of the two filters are given below for your reference only, as you don't really need the equation for your analysis in this problem:

$$h_{hp}(n) = 2(-1)^n h_{lp}(n) = 2(-1)^n \left(\frac{\sin\left(\frac{\pi}{2}n\right)}{\frac{\pi}{2}n} \right), \quad -\infty < n < \infty$$

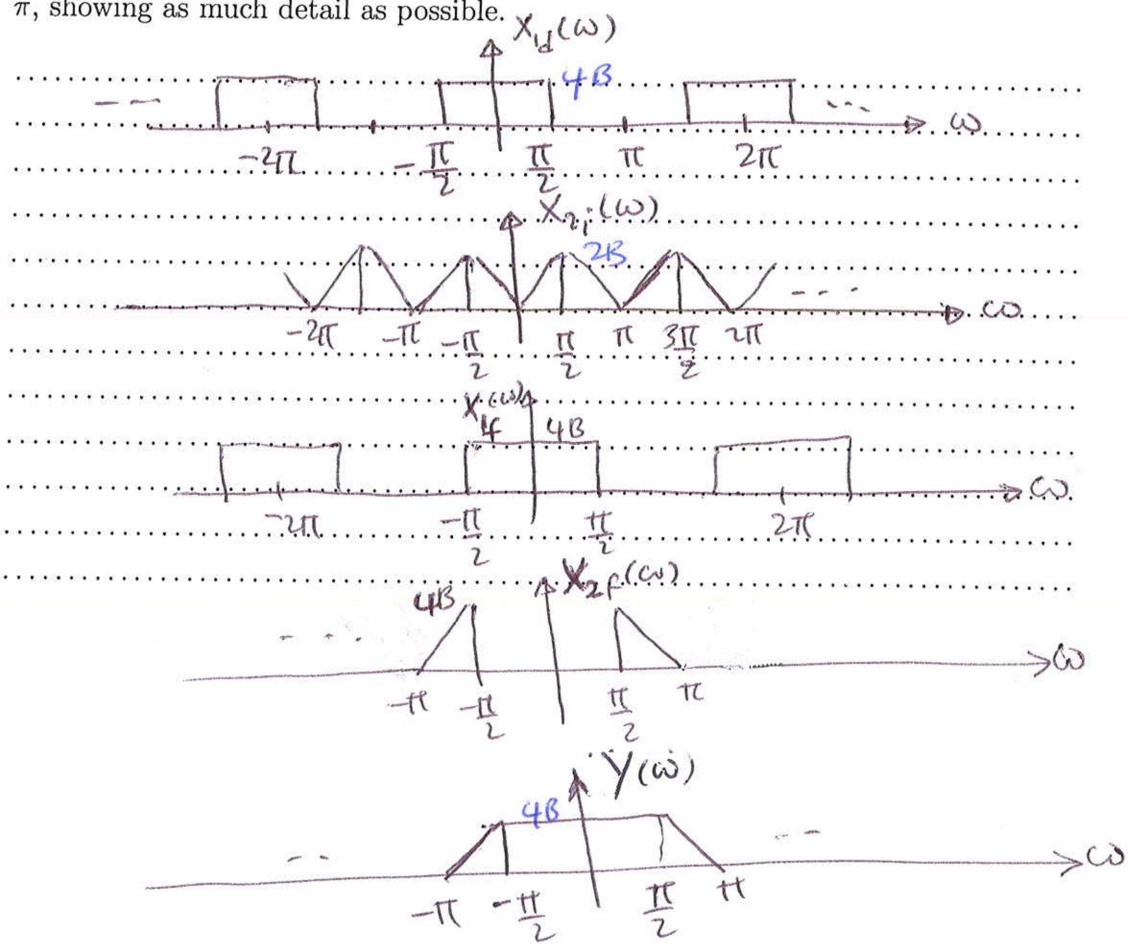
(*Hint*: You should not need to do a lot of mathematical analysis for this problem, and should be able to sktech the required DTFT magnitudes with very little analysis.)



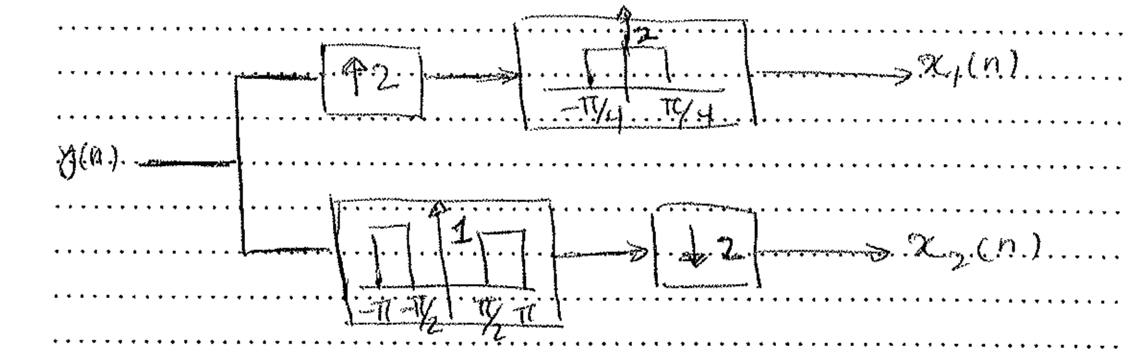
(a) (6 points) Plot the magnitude DTFT of $x_1(n)$ and $x_2(n)$ over $-2\pi < \omega < 2\pi$, showing as much detail as possible. Show the important points on the frequency axis both in terms of rad/sample as well as Hertz.



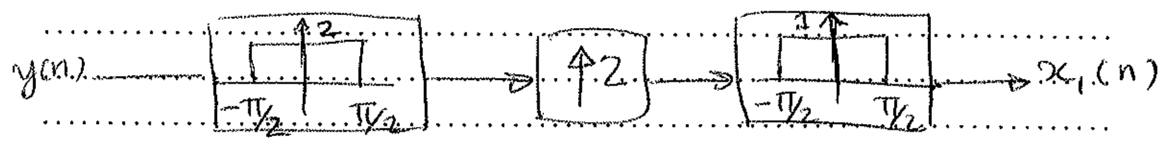
(b) (6 points) Plot the magnitude DTFT of the composite signal $y(n)$ over $-\pi < \omega < \pi$, showing as much detail as possible.



(c) (6 points) Using only downsamplers, upsamplers, and ideal filters (with proper gains), can you draw the block diagram of a system that can be used to recover the individual signals $x_1(n)$ and $x_2(n)$ from the composite signal $y(n)$?

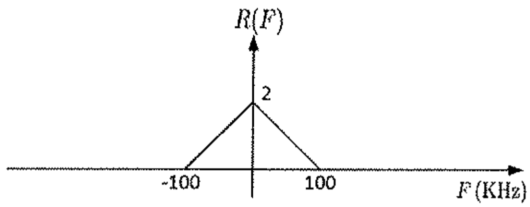


If we were to use the same $\frac{\pi}{2}$ cutoff for the top filter, we would have to implement it as:

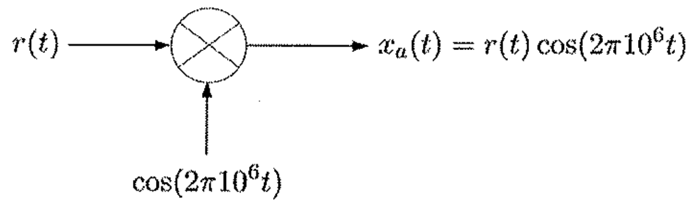


5. (25 points) **Convolution, Sampling, DTFT**:

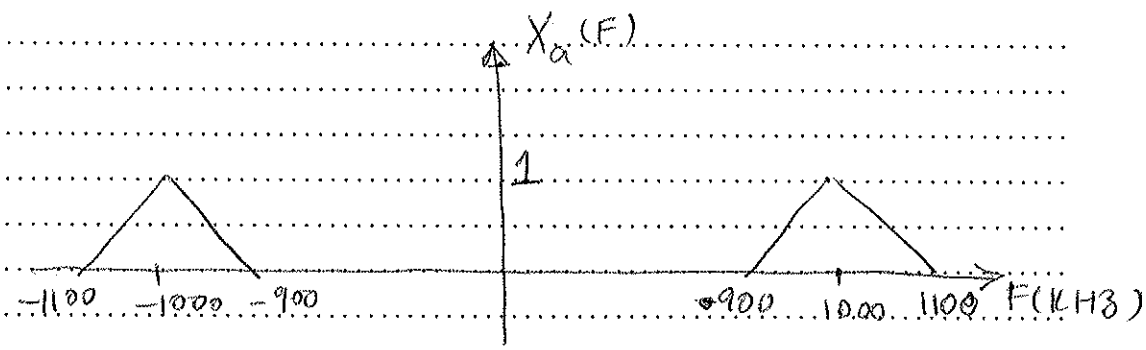
Consider an information-bearing signal, $r(t)$, with the following frequency spectrum:



The signal $r(t)$ is then modulated onto a carrier with frequency 1.0 MHz:



(a) (2 points) Sketch the frequency spectrum of $x_a(t)$ (i.e., $X_a(F)$). Please show the values for all the important points on both axes.



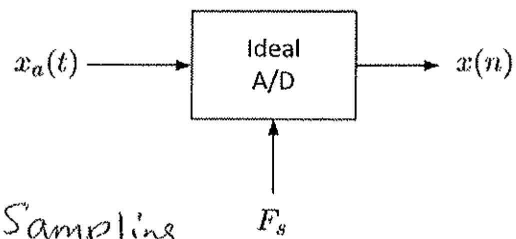
Remember that :

$$X_a(f) = R(F) * \mathcal{F}\left\{ \frac{1}{2} e^{j2\pi F_0 t} + \frac{1}{2} e^{-j2\pi F_0 t} \right\}, \quad F_0 = 1000 \text{ kHz}$$

$$= R(F) * \left[\frac{1}{2} \delta(F - F_0) + \frac{1}{2} \delta(F + F_0) \right]$$

$$= \frac{1}{2} \left[R(F - F_0) + R(F + F_0) \right]$$

(b) (4 points) Now, assume $x_a(t)$ is sampled as shown below. What would be the minimum *bandpass* sampling rate, F_s, in KHz, in order to avoid any aliasing error?



Bandpass Sampling Formula:

$\frac{2F_H}{K} \leq F_s \leq \frac{2F_L}{K-1}$ $F_H = 1100$ KHz, $F_L = 900$ KHz, $B = 200$ KHz

$K_{max} = \lfloor \frac{F_H}{B} \rfloor = \lfloor \frac{1100}{200} \rfloor = 5$

$\Rightarrow F_{S_{min}} = \frac{2F_H}{K_{max}} = \frac{2 \times 1100}{5} = 440$ KHz

For $K = 5$: $440 \leq F_S \leq 450$ KHz

$\Rightarrow F_{S_{min}} = 440$ KHz ✓

(c) (4 points) Now, assume that our sampling clock rate can have up to ±5% variation, i.e., with a nominal sampling rate, F_0, the actual sampling rate may be in the range $0.95F_0 < F_s < 1.05F_0$. What would be the *minimum nominal bandpass* sampling rate that we can select in this case in order to ensure that even with the actual sampling rate, we will not have any aliasing error?

So we need to find K_{max} such that:

$\frac{2F_H}{K} < 0.95F_0 < F_S < 1.05F_0 < \frac{2F_L}{K-1}$

$\Rightarrow \frac{\cancel{2}F_H}{K} \times \frac{1}{0.95} < \frac{\cancel{2}F_L}{K-1} \times \frac{1}{1.05} \rightarrow 0.95K \cdot F_L > 1.05(K-1)F_H$

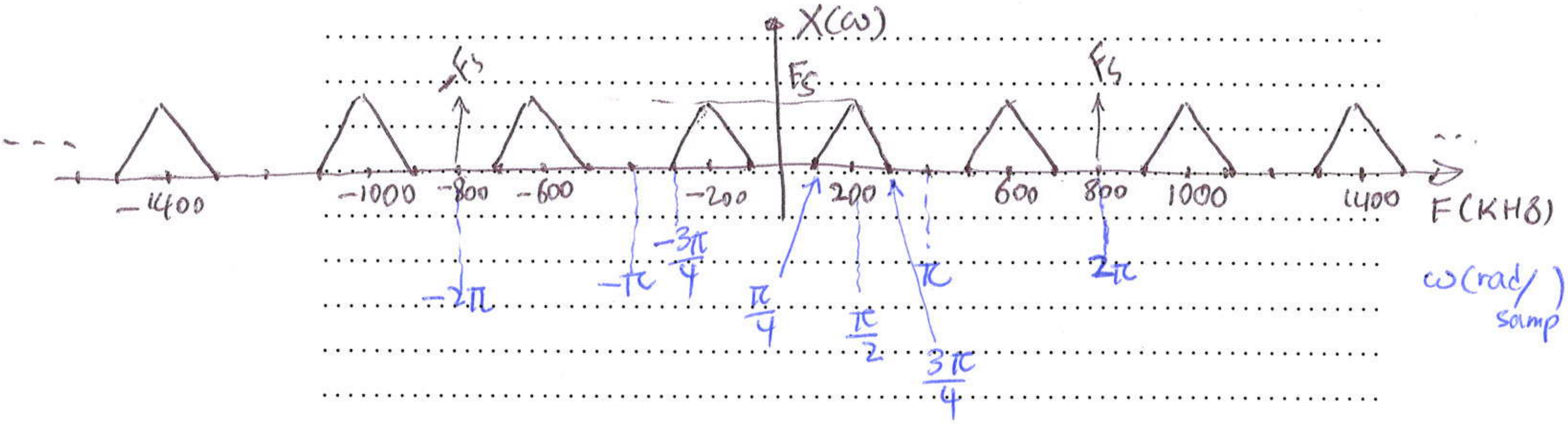
$\Rightarrow K(1.05F_H - 0.95F_L) < 1.05F_H \rightarrow$

$K < \frac{1.05 \times 1100}{1.05 \times 1100 - 0.95 \times 900} = \frac{1155}{300} = 3.85 \rightarrow K_{max} = 3$

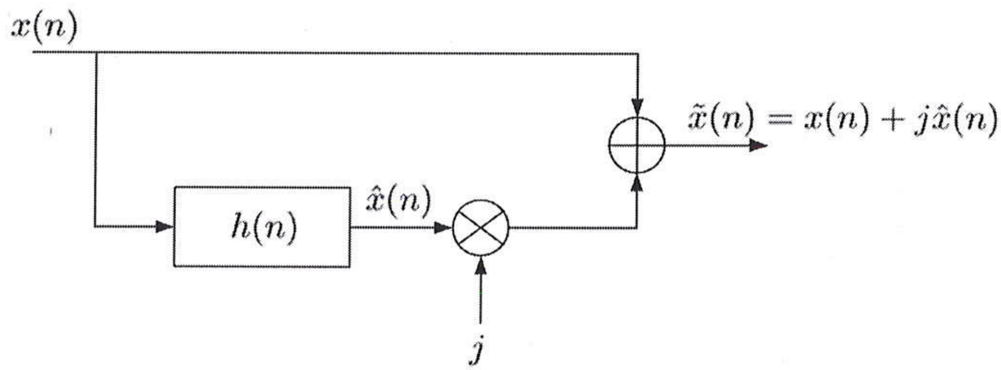
$\Rightarrow 733.3 < F_S < 900 \rightarrow F_{S_{0\,min}} = 772$ KHz $= \frac{733.3}{0.95}$

$F_{S_0\,max} \simeq 857$ KHz $= \frac{900}{1.05}$

(d) (4 points) Now, assume $F_s = 800$ KHz. Sketch the frequency spectrum, $X(\omega)$, for $-2\pi \leq \omega \leq 2\pi$. Please show the values for all the important points on both axes. Also please show values on the frequency axis in both KHz and rad/sample scales.



(e) (5 points) Now, consider the following system:

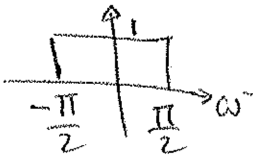


The sequence $x(n)$ from Part (d) is passed through an LTI system with the following impulse response:

$$h(n) = 2\frac{\sin^2\left(\frac{\pi}{2}n\right)}{\pi n} \tag{2}$$

Plot the magnitude and phase of the DTFT of $h(n)$, $H(\omega)$, over $-\pi < \omega < \pi$, i.e., plot both $|H(\omega)|$ and $\angle H(\omega)$ separately over $-\pi < \omega < \pi$.

Hint: $\text{DTFT}\{\frac{\sin \omega_c n}{\pi n}\} = \prod\left(\frac{\omega}{2\omega_c}\right)$



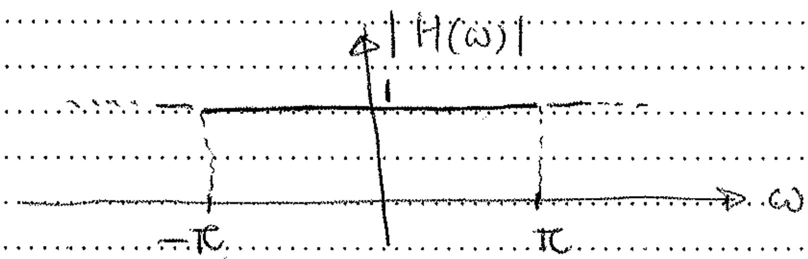
$$h(n) = 2\,\frac{\sin^2(\frac{\pi}{2}n)}{\pi n} = \frac{\sin(\frac{\pi}{2}n)}{\pi n} \times 2\sin(\tfrac{\pi}{2}n)$$

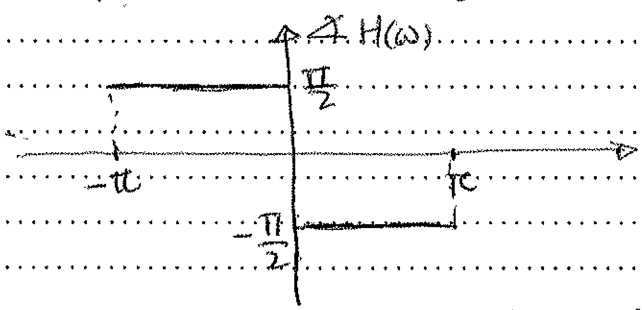
$$\Longrightarrow H(\omega) = \Pi\left(\frac{\omega}{\pi}\right) * \mathcal{F}\left\{2\times\left[\frac{1}{2j}e^{j\frac{\pi}{2}n} - \frac{1}{2j}e^{-j\frac{\pi}{2}n}\right]\right\}$$

$$\Longrightarrow H(\omega) = \Pi\left(\frac{\omega}{\pi}\right) * \left[-j\,\delta(\omega - \tfrac{\pi}{2}) + j\,\delta(\omega + \tfrac{\pi}{2})\right]$$

(Note: $\frac{1}{j} = -j$)

$$\Longrightarrow H(\omega) = -j\,\Pi\left(\frac{\omega - \pi/2}{\pi}\right) + j\,\Pi\left(\frac{\omega + \frac{\pi}{2}}{\pi}\right)$$

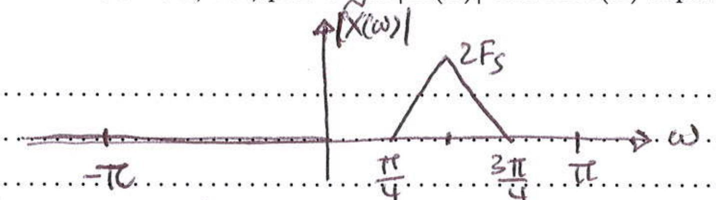




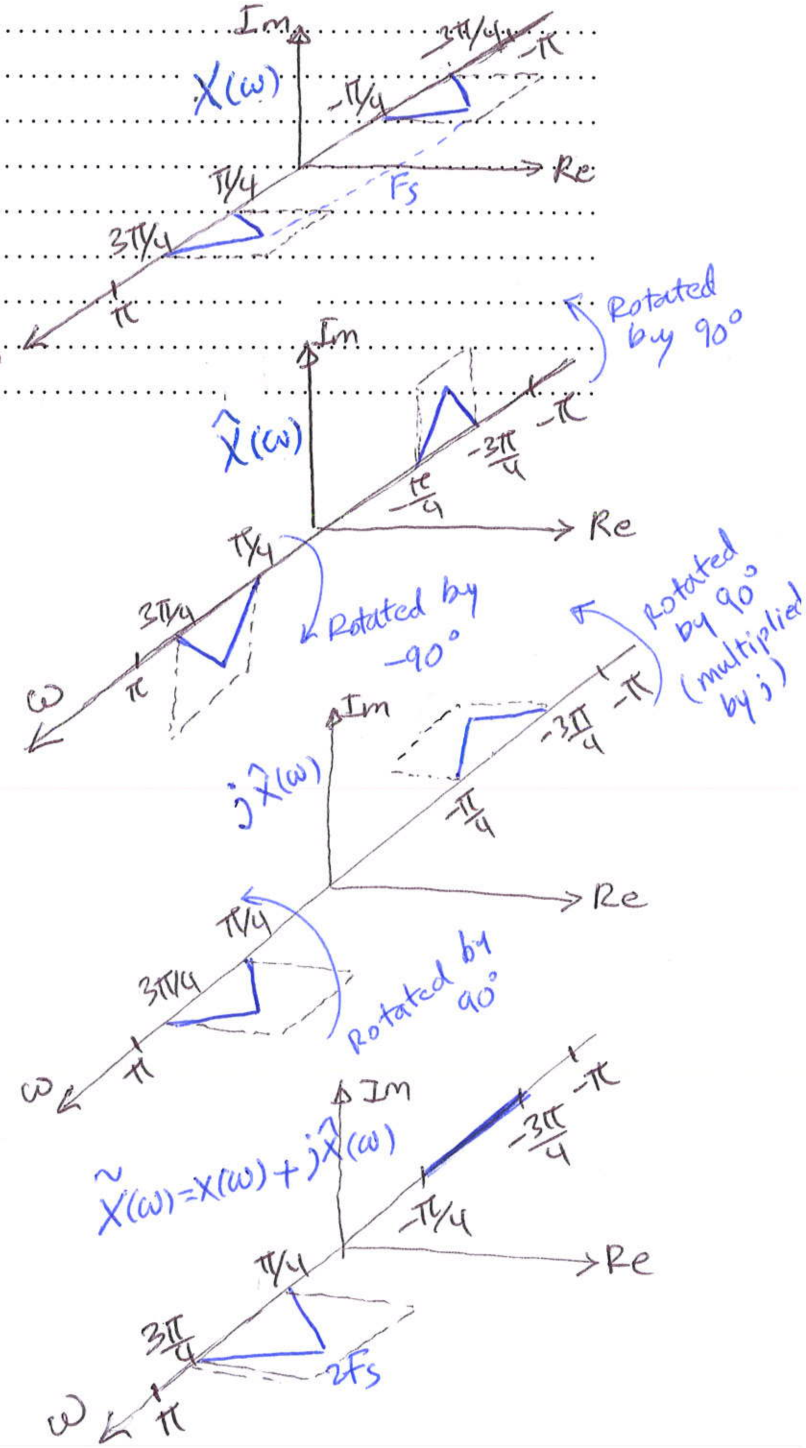
h(n) is known as the ideal discrete-time "Hilbert Transform". As you can see from H(ω), it is an all-pass filter (i.e. unity magnitude across the freq) and it rotates the phase by 90 deg for negative frequencies and by −90 deg for positive frequencies. As seen in the next part, it can be used to generate the associated complex signal of the input, keeping the spectrum only on one side, thus creating a so-called "analytic" signal. Hilbert Transforms are used in a variety of applications in DSP systems to generate complex and quadrature signals.

(f) (6 points) Plot the magnitude and phase of the DTFT of the complex-valued sequence $\tilde{x}(n)$, $\tilde{X}(\omega)$, over $-\pi < \omega < \pi$, i.e., plot both $|\tilde{X}(\omega)|$ and $\measuredangle \tilde{X}(\omega)$ separately over $-\pi < \omega < \pi$.

$\measuredangle \tilde{X}(\omega) = 0$



The plots below should make it clear how we ended up with the spectrum on the positive freq side only thanks to Hilbert transform.



6. (18 points) **Sampling, DFT, FFT**:

Sampling a continuous-time baseband signal, $x_a(t)$, over a duration of 1 second, has generated 2048 samples.

(a) (3 points) Assuming there was no aliasing, what would be the highest frequency in the spectrum of $x_a(t)$?

$F_s = 2048\ Hz \rightarrow F_{max} = \frac{F_s}{2} = 1024\ Hz$

(b) (3 points) If we obtain the 2048-point FFT of the available samples, what would be the frequency spacing in Hertz between each two adjacent frequency bins?

DFT/FFT resolution is $\frac{F_s}{N} = \frac{2048}{2048} = 1\ Hz$

(c) (3 points) Suppose we are only interested in the frequency bins within the range $200 \leq F \leq 400$ Hz. How many complex multiplications would be needed to calculate these frequency bins using the direct 2048-point DFT computation?

With direct DFT calculation, each bin requires N complex multiplication

$\Rightarrow$ Given 201 bins in $200 \leq F \leq 400$ range, # of complex mult: $201 \times 2048 = 411648$

(d) (3 points) How many complex multiplications would be needed to obtain all 2048 bins if we use *Decimation-in-Time* Radix-2 2048-point FFT algorithm instead? Compare with the number you found in Part (c) for only a subset of frequency bins.

$\frac{N}{2} \log_2 N = \frac{2048}{2} \log_2 2048 = 11264$

(e) (3 points) How many complex multiplications would be needed to obtain all 2048 bins if we use *Decimation-in-Frequency* Radix-2 2048-point FFT algorithm instead? Compare with your numbers in Part (c) and (d).

D.I.F. Radix-2 N-point FFT requires exactly the same # of complex multi: $\frac{N}{2} \log_2 N = 11264$

(f) (3 points) What would be the minimum number of required frequency bins before the Decimation-in-Time Radix-2 2048-point FFT gets to be more computationally efficient than the direct 2048-point DFT computation?

Calculating m bins w/ Direct DFT requires $m \times N$ complex multiplications $\Rightarrow$

$m \cdot N \leq \frac{N}{2} \log_2 N \rightarrow m \leq \frac{1}{2} \log_2 N = 5.5$

$\Rightarrow m_{min} = 5$. So to calculate anything more than 5 freq bins, it would be better to just get the full 2048-pt FFT!

Scratch Page