MIDTERM EXAMINATION

1. (30 PTS) **True** or **False**? Explain or give counter-examples:

(a) The ROC of $X(z)$ excludes the points at $\pm\infty$ if $x(n)$ is a causal sequence.
Solution:
False. By definition of the z-transform, $X(z) = \sum_{n=-\infty}^{+\infty} x(n)z^{-n} = \sum_{n=0}^{+\infty} x(n)z^{-n}$ where the last equality follows from the causality of $x(n)$. Since $X(z)$ does not contain any positive power of z, $\pm\infty$ can be included in the ROC.

(b) Any given $X(z)$ can be the transform of at most two possible sequences, $x(n)$.
Solution:
False. Suppose

$$X(z) = \frac{z}{z-1} + \frac{z}{z-2}$$

This can be the transform of three sequences, depending on how the ROC is defined. If $|z| > 2$, then $x(n) = u(n) + 2^n u(n)$. If $1 < |z| < 2$, then $x(n) = u(n) - 2^n u(-n-1)$. And finally, if $|z| < 1$, then $x(n) = -u(-n-1) - 2^n u(-n-1)$.

(c) If $\mathcal{Z}^+[x_1(n)] = \mathcal{Z}^+[x_2(n-1)]$, then $x_2(n) = x_1(n+1)$.
Solution:
False. Suppose $x_1(n) = u(n) + u(-n-1)$, then $\mathcal{Z}^+[x_1(n)] = \mathcal{Z}[x_1(n)u(n)] = \mathcal{Z}[u(n)] = \frac{z}{z-1}$, $|z| > 1$. Consider now $x_2(n) = u(n+1)$, then $\mathcal{Z}^+[x_2(n-1)] = \mathcal{Z}^+[u(n)] = \frac{z}{z-1}$, $|z| > 1$. Thus we have $\mathcal{Z}^+[x_1(n)] = \mathcal{Z}^+[x_2(n-1)]$ but $x_2(n) \neq x_1(n+1)$.

(d) If $\mathcal{Z}[x_1(n)] = \mathcal{Z}[x_2(n-1)]$, then $x_2(n) = x_1(n+1)$.
Solution:
False. Suppose $x_1(n) = u(n)$, then $\mathcal{Z}[x_1(n)] = \frac{z}{z-1}$, $|z| > 1$. Consider now $x_2(n) = -u(-n-2)$, then $\mathcal{Z}[x_2(n-1)] = \mathcal{Z}[-u(-n-1)] = \frac{z}{z-1}$, $|z| < 1$. Thus we have $\mathcal{Z}[x_1(n)] = \mathcal{Z}[x_2(n-1)]$ but $x_2(n) \neq x_1(n+1)$.

(e) The product of two periodic sequences is not necessarily periodic.
Solution:
False. Let us denote by N_1 the period of the first sequence and by N_2 the period of the second sequence. Then after $N_3 = LCM(N_1, N_2)$ samples, both sequences repeat themselves thus their product must also repeat itself after N_3 samples, which makes the product also periodic of period N_3.

(f) The system $y(n) = y(n-1) + x(2n),\ y(-1) = 0,\ n \geq 0$, is time-invariant.
Solution:
False. Let $x_1(n) = \delta(n)$, then by iterating we can find that the output to $x_1(n)$ is

$y_1(n) = u(n)$. Let us now consider $x_2(n) = x_1(n-1) = \delta(n-1)$. Then $x_2(2n) = x_1(2n-1) = \delta(2n-1) = 0$ for all n, thus the output to $x_2(n)$ is $y_2(n) = 0$ for all n. Since $y_2(n) \neq y_1(n-1)$, the system is time-variant.

(g) Every periodic sequence is an energy sequence.
Solution:
False. Any non-zero periodic sequence has infinite energy, and thus it is not an energy sequence.

(h) A relaxed system can be nonlinear.
Solution:
True. Consider for instance the following system: $y(n) = x^2(n)$. This is a relaxed and nonlinear system.

(i) The result of convolving two sequences is always a longer sequence.
Solution:
False. If we convolve for example $\delta(n) - \delta(n-1)$ with $u(n)$, we obtain:

$$(\delta(n) - \delta(n-1)) * u(n) = u(n) - u(n-1) = \delta(n)$$

(j) If $x(n)$ is an energy sequence then so is $x^2(-n)$.
Solution:
True. If $x(n)$ is an energy sequence then $\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$. Since $\sum_{n=-\infty}^{\infty} |x(-n)|^4 = \sum_{n=-\infty}^{\infty} |x(n)|^4 \leq \left(\sum_{n=-\infty}^{\infty} |x(n)|^2\right)^2 < \infty$, we conclude that $x^2(-n)$ is an energy sequence as well.

2. (30 PTS) Consider the block diagram representation shown in the figure consisting of the series cascade of two LTI systems, $\mathcal{S}_1$ and $\mathcal{S}_2$, with a squaring device in between. The following is known about the system:

(i) $\frac{1}{2}\delta(n-3) - \delta(2n-4) = \mathcal{S}_1[\delta(n-1)]$.
(ii) $y(n) = (1/3)^{n-1}u(2n-1)$ when $x(n) = \delta(n)$.

Determine a difference equation relating $y(n)$ and $x(n)$.

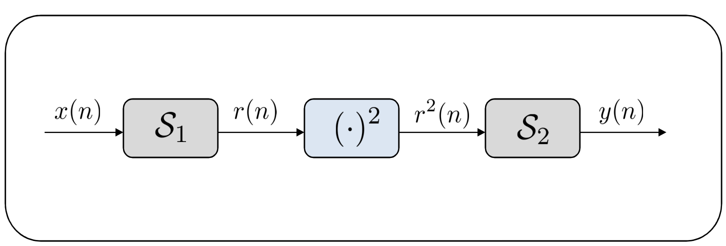

Figure 1: A cascade of three systems.

Solution:
Let us first find the impulse response $h_1(n)$ of the first system $\mathcal{S}_1$. We know that $\mathcal{S}_1[\delta(n-1)] =$

$\frac{1}{2}\delta(n-3) - \delta(2n-4) = \frac{1}{2}\delta(n-3) - \delta(n-2)$, therefore using the properties of LTI system, we conclude that: $h_1(n) = \mathcal{S}_1[\delta(n)] = \frac{1}{2}\delta(n-2) - \delta(n-1)$. Thus $r(n)$ and $x(n)$ are related as follows: $r(n) = \frac{1}{2}x(n-2) - x(n-1)$.

Let us now find the impulse response of the second system $\mathcal{S}_2$. When $\delta(n)$ is applied as an input to the first system, the output is then $r(n) = \frac{1}{2}\delta(n) - \delta(n-1)$. Thus $r^2(n) = \frac{1}{4}\delta(n-2) + \delta(n-1)$. Let us denote by $z(n)$ the input to the second system. Thus when $z(n) = r^2(n) = \frac{1}{4}\delta(n-2) + \delta(n-1)$, the corresponding output is $y(n) = (1/3)^{n-1}u(2n-1) = (1/3)^{n-1}u(n-1)$. Using the z-transform, we can now obtain the transfer fucntion of the second system $H_2(z)$:

$$
\begin{aligned}
H_2(z) &= \frac{Y(z)}{Z(z)} \\
&= \frac{(z-\frac{1}{3})^{-1}}{\frac{1}{4}z^{-2} + z^{-1}} \\
&= \frac{z^2}{(z-\frac{1}{3})(z+\frac{1}{4})} \\
&= \frac{z^2}{z^2 - \frac{1}{12}z - \frac{1}{12}} \\
&= \frac{1}{1 - \frac{1}{12}z^{-1} - \frac{1}{12}z^{-2}}
\end{aligned}
$$

Therefore, system $\mathcal{S}_2$ can be described by

$$y(n) - \frac{1}{12}y(n-1) - \frac{1}{12}y(n-2) = z(n) = r^2(n)$$

Since $r(n) = \frac{1}{2}x(n-2) - x(n-1)$, the difference equation that relates $y(n)$ to $x(n)$ is as follows:

$$y(n) - \frac{1}{12}y(n-1) - \frac{1}{12}y(n-2) = \left(\frac{1}{2}x(n-2) - x(n-1)\right)^2$$

3. (20 PTS) Consider the difference equation:

$$y(n) - \frac{1}{3}y(n-1) \;=\; n\left(\frac{1}{4}\right)^{n/2} u(2n-1), \quad n \geq 0, \;\; y(-1) = 2.$$

   (a) Use the $z-$transform technique to determine the zero-state response of the system.

   Solution:

   We can equivalently write the input as follows:

$$n\left(\frac{1}{4}\right)^{n/2} u(2n-1) = n\left(\frac{1}{4}\right)^{n/2} u(n-1) = n\left(\frac{1}{2}\right)^{n} u(n)$$

   Thus, the z-transform of $y_{zs}(n)$ satisfies the following equation:

$$Y_{zs}(z) - \frac{1}{3}z^{-1}Y_{zs}(z) = \frac{1}{2}\frac{z}{\left(z-\frac{1}{2}\right)^2}$$

Thus,

$$Y_{zs}(z) = \frac{1}{2}\frac{z^2}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)^2}, \quad |z| > \frac{1}{2}$$

We can now perform the partial expansion of $Y_{zs}(z)$:

$$Y_{zs}(z) = \frac{A}{z-\frac{1}{3}} + \frac{B}{\left(z-\frac{1}{2}\right)^2} + \frac{C}{z-\frac{1}{2}}$$

where A,B and C are found using:

$$A = Y_{zs}(z)(z-\frac{1}{3})|_{z=\frac{1}{3}} = 2$$
$$B = Y_{zs}(z)(z-\frac{1}{2})^2|_{z=\frac{1}{2}} = \frac{3}{4}$$
$$C = \frac{d}{dz}\left(Y_{zs}(z)(z-\frac{1}{2})^2\right)|_{z=\frac{1}{2}} = -\frac{3}{2}$$

which yields

$$Y_{zs}(z) = \frac{2}{z-\frac{1}{3}} + \frac{3/4}{\left(z-\frac{1}{2}\right)^2} - \frac{3/2}{z-\frac{1}{2}}$$

Therefore,

$$y_{zs}(n) = \left(2\left(\frac{1}{3}\right)^{n-1} + \frac{3}{2}(n-1)\left(\frac{1}{2}\right)^{n-1} - \frac{3}{2}\left(\frac{1}{2}\right)^{n-1}\right)u(n-1)$$

(b) Without redoing the calculations, how would your answer to part (a) change if the sequence on the right-hand side is replaced by $(n-1)\left(\frac{1}{4}\right)^{n/2}u(n-2)$?
Solution:
Since $(n-1)\left(\frac{1}{4}\right)^{n/2}u(n-2) = (n-1)\left(\frac{1}{2}\right)^{n}u(n-2) = (n-1)\left(\frac{1}{2}\right)^{n}u(n-1) = \frac{1}{2}(n-1)\left(\frac{1}{2}\right)^{n-1}u(n-1)$, the corresponding zero-state response is thus $\frac{1}{2}y_{zs}(n-1)$.

4. (20 PTS) Consider the running weighted average system

$$y(n) = \frac{1}{S_n}\sum_{k=1}^{n} k\lambda^{n-k}x(k), \quad n \geq 1$$
$$y(0) = 0$$

where $\lambda \in (0,1)$ and S_n is the normalization factor defined as

$$S_n = \sum_{k=1}^{n} k\lambda^{n-k}, \quad n \geq 1$$

(a) How is S_n related to S_{n-1}?
Solution:
S_{n-1} is given by

$$S_{n-1} = \sum_{k=1}^{n-1} k\lambda^{n-1-k} = \frac{1}{\lambda}\sum_{k=1}^{n-1} k\lambda^{n-k}$$

Thus,

$$S_n = \sum_{k=1}^{n} k\lambda^{n-k} = \sum_{k=1}^{n-1} k\lambda^{n-k} + n\lambda^0 = \sum_{k=1}^{n-1} k\lambda^{n-k} + n = \lambda S_{n-1} + n$$

(b) Find a difference equation relating $y(n)$ to $y(n-1)$ and $x(n)$ in terms of S_n.
Solution:
$y(n-1)$ is given by

$$y(n-1) = \frac{1}{S_{n-1}} \sum_{k=1}^{n-1} k\lambda^{n-1-k} x(k) = \frac{1}{\lambda S_{n-1}} \sum_{k=1}^{n-1} k\lambda^{n-k} x(k)$$

Thus,

$$y(n) = \frac{1}{S_n} \sum_{k=1}^{n} k\lambda^{n-k} x(k) = \frac{1}{S_n} \sum_{k=1}^{n-1} k\lambda^{n-k} x(k) + \frac{n}{S_n}\lambda^0 x(n) = \frac{\lambda S_{n-1}}{S_n} y(n-1) + \frac{n}{S_n} x(n)$$

Using the relation obtained in (a), $y(n)$ is related to $y(n-1)$ and $x(n)$ as follows:

$$y(n) = \frac{S_n - n}{S_n} y(n-1) + \frac{n}{S_n} x(n) = \left(1 - \frac{n}{S_n}\right) y(n-1) + \frac{n}{S_n} x(n), \quad n \geq 1$$
$$y(0) = 0$$

(c) Is the system causal? linear? stable?
Solution:

- The system is causal, since $y(n)$ depends on the present and past values of $x(n)$.
- The system is linear. Suppose we have the following two sequences $x_1(n)$ and $x_2(n)$ with their corresponding outputs $y_1(n)$ and $y_2(n)$. Let us now consider a third sequence $x_3(n) = ax_1(n) + bx_2(n)$ where a and b are scalar constants. Then the corresponding output $y_3(n)$ is

$$\begin{aligned} y_3(n) &= \frac{1}{S_n} \sum_{k=1}^{n} k\lambda^{n-k} x_3(k) = \frac{1}{S_n} \sum_{k=1}^{n} k\lambda^{n-k} \left(ax_1(k) + bx_2(k)\right) \\ &= \frac{1}{S_n} \sum_{k=1}^{n} ak\lambda^{n-k} x_1(k) + \frac{1}{S_n} \sum_{k=1}^{n} bk\lambda^{n-k} x_2(k) \\ &= ay_1(n) + by_2(n) \end{aligned}$$

  Thus the system is linear.
- The system is stable. Suppose $x(n)$ is a bounded sequence, i.e., there exists a scalar $B > 0$ such that $|x(n)| \leq B$ for all n, then

$$|y(n)| = \left| \frac{1}{S_n} \sum_{k=1}^{n} k\lambda^{n-k} x(k) \right| \leq \frac{1}{S_n} \sum_{k=1}^{n} k\lambda^{n-k} |x(k)| \leq \frac{1}{S_n} \left(\sum_{k=1}^{n} k\lambda^{n-k} \right) B = B$$

  Thus $y(n)$ is bounded and the system is stable.

(d) Is the system relaxed? time-invariant?

Solution:

- The system is relaxed, since $y(n)$ is zero as long as the input $x(n)$ is zero.
- The system is not time-invariant. Consider the following sequence $x_1(n) = x(n-1)$ and let us denote by $y_1(n)$ its corresponding output. Since $x(n)$ is considered as a permissible input for $n \geq 1$, we can consider $x(0) = 0$ so that $x_1(1) = x(0) = 0$ which implies that $y_1(1) = 0$. Then for any $n \geq 2$

$$y_1(n) = \frac{1}{S_n} \sum_{k=1}^{n} k\lambda^{n-k} x_1(k) = \frac{1}{S_n} \sum_{k=2}^{n} k\lambda^{n-k} x_1(k) = \frac{1}{S_n} \sum_{k=2}^{n} k\lambda^{n-k} x(k-1)$$

Let us use the following change of variables $\ell = k - 1$, then

$$y_1(n) = \frac{1}{S_n} \sum_{\ell=1}^{n-1} (\ell + 1)\, \lambda^{n-\ell-1} x(\ell), \quad n \geq 2$$

$$y_1(1) = 0$$

Let us now compare $y_1(n)$ with $z(n) = y(n-1)$. $z(n)$ is given by:

$$z(n) = y(n-1) = \frac{1}{S_{n-1}} \sum_{\ell=1}^{n-1} \ell \lambda^{n-\ell-1} x(\ell), \quad n \geq 2$$

$$z(1) = 0$$

$y_1(n) \neq z(n)$, which implies that the system is not time-invariant.