MIDTERM EXAMINATION

1. (50 PTS) **True** or **False**? Explain or give counter-examples:

   (a) If a system is unstable, then it cannot be FIR.

   (b) The system $y(n) = y(n-1) + x^2(n)$, $y(-1) = 0$, $n \geq 0$ is time-variant.

   (c) If two systems are relaxed, then their series cascade is also relaxed.

   (d) If an LTI system is stable, then its impulse response sequence is an energy sequence.

   (e) Every periodic sequence is a power sequence.

   (f) There cannot exist a sequence that has both infinite power and infinite energy.

   (g) The convolution of two causal sequences can be a non-causal sequence.

   (h) The complete solution of an LTI difference equation coincides with its zero-state solution.

   (i) The steady-state response of a stable LTI system is the zero sequence.

   (j) The zero-input solution of a relaxed system is the zero sequence.

2. (15 PTS) A class has N students numbered 1 through N. Let $x(n)$ denote the grade of student n in the midterm examination on a scale of 100. Let $y(n)$ denote the maximum grade of the first n students (i.e., of students $1, 2, \ldots, n$).

   (a) Find a description for the system that relates $y(n)$ to $y(n-1)$ and $x(n)$ for $n \geq 1$.

   (b) Is the system causal? linear? stable?

   (c) Assume $x(n) = 50$ for all students. What is the response of the system?

3. (25 PTS) A causal system is described by the following second-order difference equation:

$$y(n) + \frac{1}{12}y(n-1) - \frac{1}{12}y(n-2) = x^2(n), \quad y(-1) = 0, \quad y(-2) = -12, \quad n \geq 0$$

   where $x(n)$ denotes causal input sequences.

   (a) Is the system relaxed? Why or why not?

   (b) Is the system linear? Why or why not?

   (c) Determine the resulting sequence $y(n)$ when $x(n) = \left(\frac{1}{2}\right)^n u(n)$.

   (d) Determine the $z-$transform and ROC of $y(n)$ from part (c).

   (e) What would the response $y(n)$ be if the initial conditions are each multiplied by 14 and the input sequence of part (c) is multiplied by 7?

4. (10 PTS) Consider the causal LTI system

$$y(n) = \frac{1}{2}y(n-1) + \left(\frac{1}{3}\right)^{2n} \cos(\pi n) u(n-2), \quad n \geq 0$$

   Use the $z-$transform technique to determine $y(n)$.

SOLUTIONS FOR MIDTERM EXAMINATION

1. (50 PTS) **True** or **False**? Explain or give counter-examples:

   (a) If a system is unstable, then it cannot be FIR.

   Solution: True. It is equivalent to prove that an FIR system is always stable. Suppose we have a bounded input $|x(n)| \leq B$ for all n. The output of an FIR LTI system

   $$y(n) \triangleq \sum_{k=0}^{N} a_k x(n-k)$$

   is bounded since

   $$|y(n)| \leq \sum_{l=0}^{N} |a_k| \cdot |x(n-k)| \leq B \sum_{k=0}^{N} |a_k| \triangleq B' \leq \infty$$

   (b) The system $y(n) = y(n-1) + x^2(n), y(-1) = 0, n \geq 0$ is time-variant.

   Solution: False. The overall system can be regarded as a series cascade of two systems. The first system is $x'(n) = x^2(n), n \geq 0$, which is time-invariant. The second system is $y(n) = y(n-1) + x'(n), y(-1) = 0, n \geq 0$, which is LTI. Therefore, the overall system is time-invariant since the series cascade of time-invariant systems is time-invariant.

   (c) If two systems are relaxed, then their series cascade is also relaxed.

   Solution: True. For the first relaxed system with input $x(n)$ and output $x'(n)$, we have that for any n_0,

   $$\text{if } x(n) = 0 \text{ for } n < n_0 \text{ then } x'(n) = 0 \text{ for } n < n_0$$

   The sequence $x'(n)$ is then fed into the second relaxed system with output $y(n)$ so that for any n_0

   $$\text{if } x'(n) = 0 \text{ for } n < n_0 \text{ then } y(n) = 0 \text{ for } n < n_0$$

   Combining the above results, we can conclude that for any n_0

   $$\text{if } x(n) = 0 \text{ for } n < n_0 \text{ then } y(n) = 0 \text{ for } n < n_0$$

   which is a relaxed system.

   (d) If an LTI system is stable, then its impulse response sequence is an energy sequence.

   Solution: True. For a stable LTI system, the impulse response $h(n)$ is absolutely summable, i.e., we have

   $$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \quad \Rightarrow \quad \left(\sum_{n=-\infty}^{\infty} |h(n)| \right)^2 < \infty$$

Since $|h(n)|$ is nonnegative for any n, we have

$$\sum_{n=-\infty}^{\infty} |h(n)|^2 \leq \left(\sum_{n=-\infty}^{\infty} |h(n)| \right)^2 < \infty$$

and, therefore, $h(n)$ is an energy sequence.

(e) Every periodic sequence is a power sequence.

Solution: False. If the samples of periodic sequences have infinite values, then the statement is not true. For example, the sequence $x(n) = \tan(n\pi/2)$ is periodic but not a power sequence. On the other hand, if the samples are all finite, we can prove that the statement is true. Consider a periodic sequence $x(n)$ with period N_x. Let us denote the energy within one period of $x(n)$ as

$$E_x = \sum_{k=1}^{N_x} |x(k + mN_x)|^2$$

for any integer m. Then, the average power of $x(n)$ can be expressed as

$$P_x = \lim_{N \to \infty} \left(\frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \right)$$

$$= \lim_{k \to \infty} \left(\frac{1}{2kN_x+1} \sum_{n=-kN_x}^{kN_x} |x(n)|^2 \right)$$

$$= \lim_{k \to \infty} \frac{2kE_x + |x(0)^2|}{2kN_x + 1}$$

$$= \frac{E_x}{N_x}$$

where we define $N = kN_x$ and let k go to infinity. Therefore, $x(n)$ is a power sequence.

(f) There cannot exist a sequence that has both infinite power and infinite energy.

Solution: False. Consider the sequence $x(n) = \alpha^n u(n), \alpha > 1$ which has both infinite power and infinite energy.

(g) The convolution of two causal sequences can be a non-causal sequence.

Solution: False. The ROC of a causal sequence is the exterior of a disc. Let us consider two causal sequences $x_1(n)$ and $x_2(n)$ with the ROC $= \{|z| > r_1\}$ and the ROC $= \{|z| > r_2\}$, respectively. The convolution $y(n) = x_1(n) \star x_2(n)$ has the ROC $= \{|z| > \max(r_1, r_2)\}$, which implies that $y(n)$ is also a causal sequence. Another way to see this is to use the definition of the convolution. Suppose that we can write $x_1(n) = x_1'(n)u(n - m_1)$ and $x_2(n) = x_2'(n)u(n - m_2)$ for some sequences $x_1'(n)$ and $x_2'(n)$ and some integers m_1 and m_2. Then, the convolution $y(n) = x_1(n) \star x_2(n)$ is given by

$$y(n) = \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)$$

$$= \sum_{k=m_1}^{n-m_2} x_1'(k)x_2'(n-k)u(k-m_1)u(n-k-m_2)$$

$$= \left(\sum_{k=m_1}^{n-m_2} x_1'(k) x_2'(n-k) \right) u(n - m_1 - m_2)$$

which is a causal sequence.

(h) The complete solution of an LTI difference equation coincides with its zero-state solution.

Solution: True. The complete solution can be written as $y(n) = y_{zi}(n) + y_{zs}(n)$. An LTI difference equation implies that the initial conditions are zero. By setting the input $x(n) = 0$, we get $y_{zi}(n) = 0$ and thus $y(n) = y_{zs}(n)$.

(i) The steady-state response of a stable LTI system is the zero sequence.

Solution: False. Consider an stable LTI system with $h(n) = \delta(n)$. When the input sequence is $x(n) = \cos(w_o n) u(n)$, the steady-state response of the system is $y_{zs}(n) = \cos(w_o n)$ which is not a zero sequence.

(j) The zero-input solution of a relaxed system is the zero sequence.

Solution: True. From the definition of a relaxed system, we have that for any n_0,

$$\text{if } x(n) = 0 \text{ for } n < n_0 \text{ then } y(n) = 0 \text{ for } n < n_0$$

For the input $x(n) = 0$, we always get the output $y(n) = 0$.

2. (15 PTS) A class has N students numbered 1 through N. Let $x(n)$ denote the grade of student n in the midterm examination on a scale of 100. Let $y(n)$ denote the maximum grade of the first n students (i.e., of students $1, 2, ..., n$).

(a) Find a description for the system that relates $y(n)$ to $y(n-1)$ and $x(n)$ for $n \geq 1$.

Solution:

$$y(n) = \max_{1 \leq k \leq n} x(k)$$

$$= \max \left\{ \max_{1 \leq k \leq n-1} x(k), x(n) \right\}$$

$$= \max \left\{ y(n-1), x(n) \right\}$$

(b) Is the system causal? linear? stable?

Solution: The system is causal because $y(n)$ does not depend on future values of $x(m)$ for any $m > n$.

The system is non-linear. To see this, let us consider two different input sequences $x_1(n)$ and $x_2(n)$, and denote their corresponding output sequences by

$$y_1(n) = \max_{1 \leq k \leq n} x_1(k), \qquad y_2(n) = \max_{1 \leq k \leq n} x_2(k)$$

If we have the input sequence $x(n) = \alpha x_1(n) + \beta x_2(n)$ for any α and β, we have the following result in general:

$$y(n) = \max_{1 \leq k \leq n} x(k)$$

$$= \max_{1 \leq k \leq n} \alpha x_1(k) + \beta x_2(k)$$

$$\neq \alpha \max_{1 \leq k \leq n} x_1(k) + \beta \max_{1 \leq k \leq n} x_2(k)$$

$$= \alpha y_1(n) + \beta y_2(n)$$

The system is stable because with any bounded input sequence $|x(n)| \leq 100$ for all $n \geq 1$, we always get a bounded output sequence $|y(n)| \leq 100$.

(c) Assume $x(n) = 50$ for all students. What is the response of the system?
Solution: From the result in (a) we have $y(n) = 50$ for all $n \geq 1$.

3. (25 PTS) A causal system is described by the following second-order difference equation:

$$y(n) + \frac{1}{12}y(n-1) - \frac{1}{12}y(n-2) = x^2(n), \quad y(-1) = 0, \quad y(-2) = -12, \quad n \geq 0$$

where $x(n)$ denotes causal input sequences.

(a) Is the system relaxed? Why or why not?
Solution: No, the system is not relaxed because the initial conditions are not zero.

(b) Is the system linear? Why or why not?
Solution: No, the system is non-linear because the initial conditions are not zero.

(c) Determine the resulting sequence $y(n)$ when $x(n) = \left(\frac{1}{2}\right)^n u(n)$.
Solution: If we introduce a new input sequence

$$x'(n) \triangleq x^2(n) = \left(\frac{1}{4}\right)^n u(n)$$

it is then equivalent to consider the following first-order difference equation:

$$y(n) + \frac{1}{12}y(n-1) - \frac{1}{12}y(n-2) = x'(n), \quad y(-1) = 0, \quad y(-2) = -12, \quad n \geq 0$$

In this system, the output sequence $y(n)$ can be expressed $y(n) = y_{zi}(n) + y_{zs}(n)$. The characteristic function for the homogeneous equation is

$$\lambda^2 + \frac{1}{12}\lambda - \frac{1}{12} = 0$$

Solving it we get the modes of the difference equation:

$$\lambda_1 = -\frac{1}{3}, \quad \lambda_2 = \frac{1}{4}$$

Therefore, the homogeneous solution can be expressed in the following form:

$$y_{zi}(n) = C_1 \left(-\frac{1}{3}\right)^n + C_2 \left(\frac{1}{4}\right)^n, \quad n \geq 0$$

Using the initial conditions $y(-1) = 0$ and $y(-2) = -12$, we get

$$\begin{cases} y(-1) = -3C_1 + 4C_2 = 0 \\ y(-2) = 9C_1 + 16C_2 = -12 \end{cases} \implies \begin{cases} C_1 = -\frac{4}{7} \\ C_2 = -\frac{3}{7} \end{cases}$$

The zero-input response is thus

$$y_{zi}(n) = \left[-\frac{4}{7}\left(-\frac{1}{3}\right)^n - \frac{3}{7}\left(\frac{1}{4}\right)^n\right] u(n)$$

We now proceed to find the impulse response $h(n)$ for the following causal LTI system:

$$y(n) + \frac{1}{12}y(n-1) - \frac{1}{12}y(n-2) = x'(n), \qquad \text{relaxed}$$

Setting the input $x(n) = \delta(n)$ we get

$$h(n) + \frac{1}{12}h(n-1) - \frac{1}{12}h(n-2) = \delta(n)$$

Since we know that $h(n) = 0$ for $n < 0$, we can get two conditions $h(-1) = 0$ and $h(0) = 1$ to solve the homogeneous solution:

$$\begin{cases} h(-1) = -3C_1 + 4C_2 = 0 \\ h(0) = C_1 + C_2 = 1 \end{cases} \implies \begin{cases} C_1 = \frac{4}{7} \\ C_2 = \frac{3}{7} \end{cases}$$

Therefore, the impulse response is

$$h(n) = \left[\frac{4}{7}\left(-\frac{1}{3}\right)^n + \frac{3}{7}\left(\frac{1}{4}\right)^n \right] u(n)$$

The zero-state response for $x'(n)$ can be found by

$$y_{zs}(n) = h(n) \star x'(n)$$

$$= \sum_{k=-\infty}^{\infty} \left[\frac{4}{7}\left(-\frac{1}{3}\right)^k + \frac{3}{7}\left(\frac{1}{4}\right)^k \right] u(k) \left(\frac{1}{4}\right)^{n-k} u(n-k)$$

$$= \sum_{k=0}^{n} \left[\frac{4}{7}\left(-\frac{1}{3}\right)^k + \frac{3}{7}\left(\frac{1}{4}\right)^k \right] \left(\frac{1}{4}\right)^{n-k} u(n)$$

$$= \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} \left[\frac{4}{7}\left(-\frac{1}{3}\right)^k + \frac{3}{7}\left(\frac{1}{4}\right)^k \right] 4^k u(n)$$

$$= \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} \left[\frac{4}{7}\left(-\frac{4}{3}\right)^k + \frac{3}{7} \right] u(n)$$

$$= \frac{3}{7}(n+1)\left(\frac{1}{4}\right)^n u(n) + \frac{4}{7}\left(\frac{1}{4}\right)^n \frac{1-\left(-\frac{4}{3}\right)^{n+1}}{1+\frac{4}{3}} u(n)$$

$$= \frac{3}{7}(n+1)\left(\frac{1}{4}\right)^n u(n) + \frac{12}{49}\left(\frac{1}{4}\right)^n u(n) + \frac{16}{49}\left(-\frac{1}{3}\right)^n u(n)$$

$$= \frac{3}{7}n\left(\frac{1}{4}\right)^n u(n) + \frac{33}{49}\left(\frac{1}{4}\right)^n u(n) + \frac{16}{49}\left(-\frac{1}{3}\right)^n u(n)$$

The complete solution is then obtained as

$$y(n) = y_{zi}(n) + y_{zs}(n) = \frac{3}{7}n\left(\frac{1}{4}\right)^n u(n) + \frac{12}{49}\left(\frac{1}{4}\right)^n u(n) - \frac{12}{49}\left(-\frac{1}{3}\right)^n u(n)$$

(d) Determine the z-transform and ROC of $y(n)$ from part (c).
Solution: The z-transform of $y(n)$ is

$$Y(z) = \frac{3}{28} \frac{z}{\left(z - \frac{1}{4}\right)^2} + \frac{12}{49} \frac{z}{z - \frac{1}{4}} - \frac{12}{49} \frac{z}{z + \frac{1}{3}}$$

with the ROC as

$$\text{ROC:} \quad \left\{ |z| > \frac{1}{3} \right\} \cap \left\{ |z| > \frac{1}{4} \right\} = \left\{ |z| > \frac{1}{3} \right\}$$

(e) What would the response $y(n)$ be if the initial conditions are each multiplied by 14 and the input sequence of part (c) is multiplied by 7?

Solution: Note that scaling $x(n)$ by 7 results in scaling $x'(n) = x^2(n)$ by 49. By the superposition properties of $y_{zi}(n)$ and $y_{zs}(n)$, the response $y(n)$ becomes

$$y_n = 14 y_{zi}(n) + 49 y_{zs}(n) = 21 n \left(\frac{1}{4} \right)^n u(n) + 27 \left(\frac{1}{4} \right)^n u(n) + 8 \left(-\frac{1}{3} \right)^n u(n)$$

4. (10 PTS) Consider the causal LTI system

$$y(n) = \frac{1}{2} y(n-1) + \left(\frac{1}{3} \right)^{2n} \cos\left(\pi n \right) u(n-2), \quad n \geq 0$$

Use the z-transform technique to determine $y(n)$.

Solution: We can rewrite the causal LTI system as the following relaxed difference equation:

$$y(n) = \frac{1}{2} y(n-1) + x(n), \quad n \geq 0$$

with the input sequence $x(n)$ as

$$
\begin{aligned}
x(n) &= \left(\frac{1}{3} \right)^{2n} \cos\left(\pi n \right) u(n-2) \\
&= \left(\frac{1}{9} \right)^n (-1)^n u(n-2) \\
&= \left(-\frac{1}{9} \right)^n u(n-2) \\
&= \left(\frac{1}{9} \right)^2 \left(-\frac{1}{9} \right)^{n-2} u(n-2)
\end{aligned}
$$

The z-transform of $x(n)$ is

$$X(z) = \left(\frac{1}{9} \right)^2 z^{-2} \frac{z}{z + \frac{1}{9}} = \left(\frac{1}{9} \right)^2 \frac{z^{-1}}{z + \frac{1}{9}}, \quad |z| > \frac{1}{9}$$

By taking the z-transform on both sides of the difference equation we get

$$Y(z) - \frac{1}{2} z^{-1} Y(z) = X(z)$$

Then, we have

$$
\begin{aligned}
Y(z) &= \frac{X(z)}{1 - \frac{1}{2} z^{-1}} \\
&= \frac{z X(z)}{z - \frac{1}{2}} \\
&= \left(\frac{1}{9} \right)^2 \frac{1}{\left(z - \frac{1}{2} \right) \left(z + \frac{1}{9} \right)}
\end{aligned}
$$

Note that since the input $x(n)$ is a causal sequence, the output $y(n)$ is also a causal sequence. Therefore, the ROC of $Y(z)$ is $|z| > \frac{1}{2}$. To find $y(n)$, we note that

$$
\begin{aligned}
Y(z) &= \left(\frac{1}{9}\right)^2 \frac{1}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{9}\right)} \\
&= \left(\frac{1}{9}\right)^2 \left(\frac{\frac{18}{11}}{z - \frac{1}{2}} - \frac{\frac{18}{11}}{z + \frac{1}{9}}\right) \\
&= \frac{2}{99}\left(\frac{1}{z - \frac{1}{2}} - \frac{1}{z + \frac{1}{9}}\right)
\end{aligned}
$$

The output sequence $y(n)$ is then given by

$$
y(n) = \frac{2}{99}\left[\left(\frac{1}{2}\right)^{n-1} - \left(-\frac{1}{9}\right)^{n-1}\right] u(n-1)
$$