EE113: Digital Signal Processing

Prof. A. H. Sayed — Oct. 28, 2003

MIDTERM EXAMINATION

(Closed Book)

1. A relaxed system is described by the difference equation

$$y(n) - \frac{1}{2}y(n-1) = x^2(n), \quad n \geq 0$$

where $x(n)$ denotes the input sequence and $y(n)$ denotes the output sequence. Prove or give counter-examples:

   (a) (10 PTS) Is the system linear?

   (b) (10 PTS) Is the system time-invariant?

   (c) (5 PTS) Is the system causal?

   (d) (10 PTS) Is the system BIBO stable?

   (e) (5 PTS) How would your answers to (a)-(d) change if the interval $n \geq 0$ were replaced by $n \leq 0$?

2. A causal system is described by the difference equation

$$y(n) - \frac{1}{2}y(n-1) = x^2(n), \quad n \geq 0$$

with initial condition $y(-1) = 2$, and where $x(n)$ denotes the input sequence.

   (a) (5 PTS) Draw a block diagram representation of the system.

   (b) (5 PTS) Find the zero-input solution of the system.

   (c) (10 PTS) Find the zero-state solution of the system corresponding to $x(n) = (1/2)^n u(n-1)$.

   (d) (10 PTS) Find the complete solution of the system. Verify that your solution satisfies the initial condition and the difference equation.

   (e) (15 PTS) Find the $z-$transform of the sequence $nx(-n) + x^2(n-2)$. Specify its region of convergence. Find also the energy of this sequence.

   (f) (5 PTS) Plot the sequence $nx(-n) + x^2(n-2)$.

3. (10 PTS) Use the z–transform to evaluate the series

$$\sum_{n=2}^{\infty} n^2 \left(\frac{1}{2}\right)^n u(n-1)$$

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MIDTERM SOLUTIONS

1. A relaxed system is described by the difference equation

$$y(n) - \frac{1}{2}y(n-1) = x^2(n), \quad n \geq 0$$

where $x(n)$ denotes the input sequence and $y(n)$ denotes the output sequence. Prove or give counter-examples:

We can find the input–output relation of the system by iteration as follows:

$$\begin{aligned}
y(0) &= x^2(0) \\
y(1) &= \frac{1}{2}x^2(0) + x^2(1) = \frac{1}{2}\left[x^2(0) + 2x^2(1)\right] \\
y(2) &= \frac{1}{4}x^2(0) + \frac{1}{2}x^2(1) + x^2(2) = \frac{1}{4}\left[x^2(0) + 2x^2(1) + 4x^2(2)\right] \\
&\vdots \\
y(n) &= \left(\frac{1}{2}\right)^n \left[x^2(0) + 2x^2(1) + \ldots + 2^n x^2(n)\right]
\end{aligned}$$

Then the general form of $y(n)$ is

$$y(n) = \left(\frac{1}{2}\right)^n \sum_{i=0}^{n} 2^i x^2(i) , \quad n \geq 0$$

(a) (10 PTS) Is the system linear?
Let $y_1(n)$ and $y_2(n)$ denote the output sequences that correspond to an input sequences $x_1(n)$ and $x_2(n)$, respectively. Then $y_1(n)$ and $y_2(n)$ can be expressed as

$$\begin{aligned}
y_1(n) &= \mathcal{S}[x_1(n)] = \left(\frac{1}{2}\right)^n \sum_{i=0}^{n} 2^i x_1^2(i) , \quad n \geq 0 \\
y_2(n) &= \mathcal{S}[x_2(n)] = \left(\frac{1}{2}\right)^n \sum_{i=0}^{n} 2^i x_2^2(i) , \quad n \geq 0
\end{aligned}$$

Now, the output $y(n)$ that corresponds to the linear combination $ax_1(n) + bx_2(n)$ is

$$y(n) = \mathcal{S}\left[ax_1(n) + bx_2(n)\right] = \left(\frac{1}{2}\right)^n \sum_{i=0}^{n} 2^i \left(ax_1(i) + bx_2(i)\right)^2 , \quad n \geq 0$$

whereas

$$ay_1(n) + by_2(n) = a\left(\frac{1}{2}\right)^n \sum_{i=0}^{n} 2^i x_1^2(i) + b\left(\frac{1}{2}\right)^n \sum_{i=0}^{n} 2^i x_2^2(i)\ , \quad n \geq 0$$

clearly,

$$\mathcal{S}\left[ax_1(n) + bx_2(n)\right] \neq ay_1(n) + by_2(n)$$

Therefore, the system is <u>nonlinear</u>.

(b) (10 PTS) Is the system time-invariant?

The system response to the input $x(n-k)$ is given by

$$y_k(n) = \mathcal{S}\left[x(n-k)\right] = \left(\frac{1}{2}\right)^n \sum_{i=k}^{n} 2^i x^2(i-k)$$

Note that the lower limit of the sum has changed from 0 to k. The reason is that we only include the samples of $x^2(n)$ for $n \geq 0$. On the other hand the output sequence delayed by k samples, $y(n-k)$, is given by

$$y(n-k) = \left(\frac{1}{2}\right)^{n-k} \sum_{i=-k}^{n-k} 2^i x^2(i) = \left(\frac{1}{2}\right)^n \sum_{i=0}^{n} 2^i x^2(i-k)$$

clearly,

$$\mathcal{S}\left[x(n-k)\right] \neq y(n-k)$$

Therefore, the system is <u>not time-invariant</u>.

We can also show that the system is not time–invariant by using the following counter example:

Let $x(n) = \delta(n+1) + \delta(n)$. then,

For $x(n) = \delta(n+1) + \delta(n)$	For $x(n-1) = \delta(n) + \delta(n-1)$
$y(0) = 1$	$y_1(0) = 1$
$y(1) = 1/2$	$y_1(1) = 3/2$
$y(2) = 1/4$	$y_1(2) = 3/4$
$y(3) = 1/8$	$y_1(3) = 3/8$
$\vdots$	$\vdots$

Clearly, $y_1(n) \neq y(n-1)$, then we conclude that the system is not time–invariant.

(c) (5 PTS) Is the system causal?

$y(n)$ depends on the current and previous samples of the input sequence. Therefore, the system is <u>causal</u>.

(d) (10 PTS) Is the system BIBO stable?

Let $y(n)$ denote the output sequence that corresponds to any bounded input sequence $x(n)$. This means that $|x(n)|$ is less that or equal to a finite positive number M.

$$|x(n)| \leq M < \infty \quad \text{for all } n$$

Now, we need to show that the output sequence is also bounded for the system to be stable. We proceed as follows:

$$\begin{aligned} |y(n)| &= \left|\left(\frac{1}{2}\right)^n \sum_{i=0}^{n} 2^i x^2(i)\right| \leq \left(\frac{1}{2}\right)^n \sum_{i=0}^{n} 2^i \left|x^2(n)\right| \\ &\leq \left(\frac{1}{2}\right)^n M^2 \sum_{i=0}^{n} 2^i = M^2 \left(2 - \left(\frac{1}{2}\right)^n\right) \leq 2M^2 < \infty \end{aligned}$$

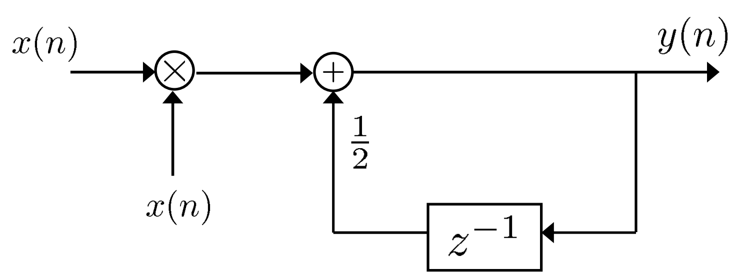
We then conclude that the system is BIBO stable.

2. A causal system is described by the difference equation

$$y(n) - \frac{1}{2}y(n-1) = x^2(n), \quad n \geq 0$$

with initial condition $y(-1) = 2$, and where $x(n)$ denotes the input sequence.

(a) (5 PTS) Draw a block diagram representation of the system.



(b) (10 PTS) Find the zero-input solution of the system.

The zero–input solution, $y_{zi}(n)$, is the solution to the homogeneous equation

$$y(n) - \frac{1}{2}y(n-1) = 0$$

subject to the y(-1) = 2. The characteristic equation of the system is

$$\lambda - \frac{1}{2} = 0 \quad \Longrightarrow \quad \lambda = \frac{1}{2}$$

Then, $y_{zi}(n) = C\left(\frac{1}{2}\right)^n$ We then use the initial condition to find C:

$$y(-1) = 2 = C\left(\frac{1}{2}\right)^{-1} \quad \Longrightarrow \quad C = 1$$

Therefore,

$$y_{zi}(n) = \left(\frac{1}{2}\right)^n \quad , \quad n \geq 0$$

(c) (10 PTS) Find the zero-state solution of the system corresponding to $x(n) = (1/2)^n u(n-1)$.

Let $z(n) = x^2(n) = \left(\frac{1}{4}\right)^n u(n-1)$, then the relaxed difference equation that we solve for the zero–state solution becomes

$$y(n) - \frac{1}{2}y(n-1) = z(n) \; , \quad y(0) = 0$$

where $y(0) = 0$ is the initial condition needed for the system to be relaxed. We can now solve for the zero–state solution in many different ways. We will do it using two different methods; first by solving as $y_{zs}(n) = y_h(n) + y_p(n)$ and then by using the z–transform approach.

(a) Using $y_{zs}(n) = y_h(n) + y_p(n)$

We can write the modified input sequence $z(n)$ as follows,

$$z(n) = x^2(n) = \left(\frac{1}{4}\right)^n u(n-1) = \frac{1}{4}\left(\frac{1}{4}\right)^{n-1} u(n-1)$$

We first find the zero–state solution to $z'(n) = \left(\frac{1}{4}\right)^n u(n)$ then we use the time invariance of the system to find the zero–state solution to $z(n)$

– Homogeneous solution $y_h(n)$ is the solution to the homogeneous equation $y(n) - \frac{1}{2}y(n-1) = 0$, that is

$$y_h(n) = C\left(\frac{1}{2}\right)^n \quad , \quad \text{for all } n$$

– Particular solution: We now use the modified difference equation

$$y(n) - \frac{1}{2}y(n-1) = z'(n) \; , \quad z'(n) = \left(\frac{1}{4}\right)^n u(n)$$

The solution has the form $y_p(n) = K\left(\frac{1}{4}\right)^n u(n)$. Substituting in the difference equation, we get

$$K\left(\frac{1}{4}\right)^n u(n) - \frac{1}{2}K\left(\frac{1}{4}\right)^{n-1} u(n-1) = \left(\frac{1}{4}\right)^n u(n)$$

Solving for K, we find that $K = -1$ for $n \geq 1$ and the particular solution is

$$y_p(n) = -\left(\frac{1}{4}\right)^n \quad , \quad n \geq 1$$

Then, the zero–state solution to $z'(n)$ is given by

$$y_{zs'} = y_h(n) + y_p(n) = C\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n \quad , \quad n \geq 1$$

Using the relaxed initial condition $y(-1) = 0$, we find that $y(0) = 1$. Using y(0), we find that $C = 2$, therefore

$$y_{zs'} = y_h(n) + y_p(n) = \left[2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right] u(n)$$

Finally, we use the time–invariance property of the system to find the zero-state response to $z(n) = \frac{1}{4}z'(n-1)$ to be

$$y_{zs}(n) = \frac{1}{4}y_{zs'}(n-1) = \left[\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right] u(n-1)$$

(b) Using z–transform
Starting from

$$y(n) - \frac{1}{2}y(n-1) = \left(\frac{1}{4}\right)^n u(n-1)$$

We take the z–transform of both sides to get

$$Y_{zs}(z) - \frac{1}{2}z^{-1} = \frac{1}{4}\frac{1}{z-1/4}$$

Then,

$$Y_{zs}(z) = \frac{1/4 \cdot z}{(z-1/2)(z-1/4)}$$

Using partial fractions, we see that $Y(z)$ can be written as

$$Y_{zs}(z) = \frac{1/2}{z-1/2} - \frac{1/4}{z-1/4}$$

Finally, we use the inverse z–transform to obtain an expression for $y_{zs}(n)$

$$y_{zs}(n) = \left(\frac{1}{2}\right)^n u(n-1) - \left(\frac{1}{4}\right)^n u(n-1)$$

(d) (10 PTS) Find the complete solution of the system. Verify that your solution satisfies the initial condition and the difference equation.

The complete solution is the sum of the zero–input solution and the zero–state solution. Then,

$$\begin{aligned} y(n) &= y_{zi}(n) + y_{zs}(n) \\ &= \left(\frac{1}{2}\right)^n u(n) + \left[\left(\frac{1}{2}\right)^n - \left(\frac{1}{2}\right)^{2n}\right] u(n-1) \end{aligned}$$

To verify that the solution satisfies the initial condition and the difference equation, we evaluate a few terms of $y(n)$ using the obtained solutions and compare them to the corresponding values that we get from iterating the difference equation. The comparison is shown below:

	Solution	Difference equation
$n=0$	$y(0)=1$	$y(0)=\frac{1}{2}y(-1)+x^2(0)=1$
$n=1$	$y(1)=\frac{3}{4}$	$y(1)=\frac{1}{2}y(0)+x^2(1)=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$
$n=2$	$y(2)=\frac{7}{16}$	$y(2)=\frac{1}{2}y(1)+x^2(2)=\frac{3}{8}+\frac{1}{16}=\frac{7}{16}$
$\vdots$	$\vdots$	$\vdots$

(e) (15 PTS) Find the $z-$transform of the sequence $nx(-n)+x^2(n-2)$. Specify its region of convergence. Find also the energy of this sequence.

Let

$$x_1(n) = nx(-n) = n\,(2)^n\,u(-n-1)$$

and

$$x_2(n) = x^2(n-2) = \left(\frac{1}{4}\right)^{n-2} u(n-3) = \frac{1}{4}\left(\frac{1}{4}\right)^{n-3} u(n-3)$$

The z–transform of $x_1(n)$ is found as follows:

$$\begin{aligned} 2^n u(-n-1) &\longleftrightarrow \frac{-\,z}{z-2}\,, \quad |z|<2 \\ n2^n u(-n-1) &\longleftrightarrow -z\frac{d}{dz}\frac{-\,z}{z-2} = \frac{-\,2z}{(z-2)^2}\,, \quad |z|<2 \end{aligned}$$

and the z–transform of $x_2(n)$ is found as follows:

$$\begin{aligned} \left(\frac{1}{4}\right)^n u(n) &\longleftrightarrow \frac{z}{z-1/4}\,, \quad |z|>\frac{1}{4} \\ \left(\frac{1}{4}\right)^{n-3} u(n-3) &\longleftrightarrow z^{-3}\frac{z}{z-1/4} = \frac{1}{z^2(z-1/4)}\,, \quad |z|>\frac{1}{4} \end{aligned}$$

Then the z–transform of $nx(-n)+x^2(n-2)$ is

$$\frac{-\,2z}{(z-2)^2} + \frac{1}{4}\frac{1}{z^2(z-1/4)}\,, \quad \frac{1}{4}<|z|<2$$

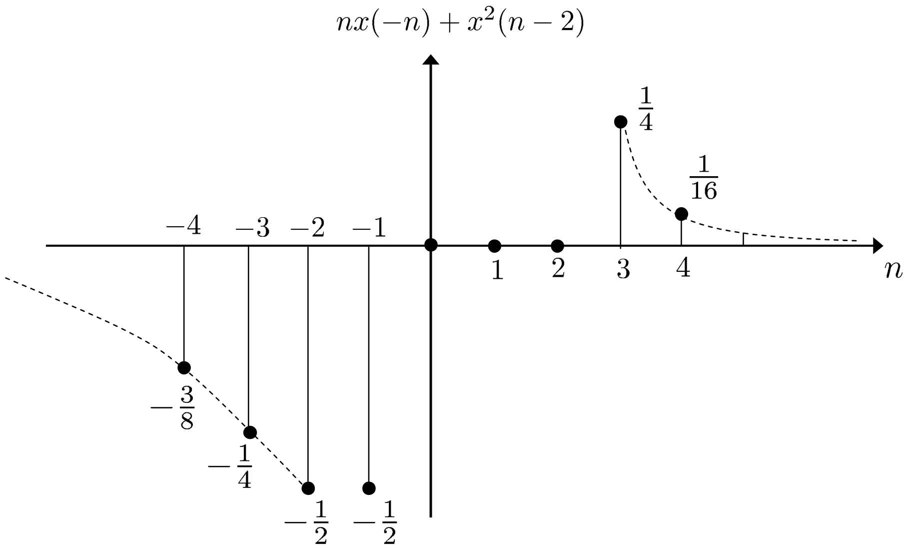
The energy of the sequence is computed as follows

$$\begin{aligned} \mathcal{E}_x &= \sum_{n=-\infty}^{\infty} |x(n)|^2 \\ &= \sum_{n=-\infty}^{\infty} \left| n\,(2)^n\,u(-n-1) + \left(\frac{1}{4}\right)^{n-2} u(n-3) \right|^2 \\ &= \sum_{n=-\infty}^{\infty} n^2\,(4)^n\,u(-n-1) + \sum_{n=-\infty}^{\infty} \left(\frac{1}{16}\right)^{n-2} u(n-3) \\ &\quad + \sum_{n=-\infty}^{\infty} n\,(2)^n \left(\frac{1}{4}\right)^{n-2} u(n-3)u(-n-1) \end{aligned}$$

Note that the third term equals to zero since $u(n-3)u(-n-1)=0$. Therefore,

$$\begin{aligned}\mathcal{E}_x &= \sum_{n=-\infty}^{-1} n^2 (4)^n + \sum_{n=3}^{\infty}\left(\frac{1}{16}\right)^{n-2} \\ &= \sum_{n=1}^{\infty} n^2 \left(\frac{1}{4}\right)^n + \sum_{n=3}^{\infty}\left(\frac{1}{16}\right)^{n-2} \\ &= \frac{20}{27} + \frac{1}{15} = \frac{109}{135} = 0.8074\end{aligned}$$

(f) (5 PTS) Plot the sequence $nx(-n) + x^2(n-2)$.



3. (10 PTS) Use the z–transform to evaluate the series

$$\sum_{n=2}^{\infty} n^2 \left(\frac{1}{2}\right)^n u(n-1)$$

$$\sum_{n=2}^{\infty} n^2 \left(\frac{1}{2}\right)^n u(n-1) = \sum_{n=2}^{\infty} n^2 (2)^{-n} u(n-1) = \sum_{n=-\infty}^{\infty} n^2 u(n-2)(2)^{-n}$$

Now let $x(n) = n^2 u(n-2)$, then

$$\sum_{n=2}^{\infty} n^2 \left(\frac{1}{2}\right)^n = \sum_{n=-\infty}^{\infty} x(n)(2)^{-n} = X(z)\,|_{z=2}$$

We then calculate X(z) as follows

$$
\begin{aligned}
u(n-2) &\longleftrightarrow z^{-2}\frac{z}{z-1} = \frac{1}{z(z-1)} \\
nu(n-2) &\longleftrightarrow -z\frac{d}{dz}\frac{1}{z(z-1)} = \frac{2z-1}{z(z-1)^2} \\
n^2u(n-2) &\longleftrightarrow -z\frac{d}{dz}\frac{2z-1}{z(z-1)^2} = \frac{4z^2-3z+1}{z(z-1)^3}
\end{aligned}
$$

Then

$$
X(z) = \frac{4z^2-3z+1}{z(z-1)^3} \qquad \Longrightarrow \qquad \sum_{2}^{\infty} n^2\left(\frac{1}{2}\right)^n = X(z)\,|_{z=2} = \frac{11}{2}
$$