EE113: Digital Signal Processing
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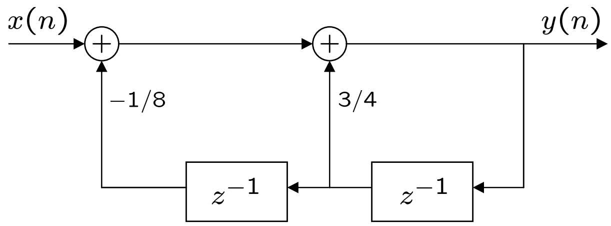
MIDTERM SOLUTIONS



1. (10 PTS) Is the system BIBO stable?
From the figure, the system difference equation is given by $y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n)$, so that its characteristic equation is:

$$\begin{aligned} \lambda^2 - \frac{3}{4}\lambda - \frac{1}{8} &= 0 \\ \left(\lambda - \frac{1}{2}\right)\left(\lambda - \frac{1}{4}\right) &= 0 \end{aligned}$$

Both modes of the system ($\lambda = \frac{1}{2}$ and $\lambda = \frac{1}{4}$) are inside the unit disc. Since the system is causal, then it is BIBO stable.

2. (30 PTS) Find the complete response of the system when $x(n) = \alpha^n u(n)$, where $|\alpha| < 1$.
Particular solution
For $x(n) = \alpha^n u(n)$, the particular solution has the form $y_p(n) = K\alpha^n u(n)$. To evaluate K, we substitute $y_p(n)$ into the difference equation

$$K\alpha^n u(n) - \frac{3}{4}K\alpha^{n-1}u(n-1) + \frac{1}{8}K\alpha^{n-2}u(n-2) = \alpha^n u(n)$$

For $n \geq 2$,

$$K\alpha^2 - \frac{3}{4}K\alpha + \frac{1}{8}K = \alpha^2$$

so that

$$K = \frac{\alpha^2}{\alpha^2 - \frac{3}{4}\alpha + \frac{1}{8}}$$

and, hence,

$$\boxed{y_p(n) = \frac{\alpha^{n+2}}{\alpha^2 - \frac{3}{4}\alpha + \frac{1}{8}}\, u(n)\ , \qquad n \geq 2}$$

Homogeneous solution

$$y_h(n) = C_1\left(\frac{1}{2}\right)^n + C_2\left(\frac{1}{4}\right)^n$$

Then, the **complete solution** $y(n)$ is given by:

$$\begin{aligned} y(n) &= y_h(n) + y_p(n) \\ &= C_1 \left(\frac{1}{2}\right)^n + C_2 \left(\frac{1}{4}\right)^n + \frac{\alpha^{n+2}}{\alpha^2 - \frac{3}{4}\alpha + \frac{1}{8}} u(n) , \qquad n \geq 2 \end{aligned}$$

Using the system difference equation and the initial conditions $y(-2) = 0$ and $y(-1) = -\frac{4}{3}$, we get

$$\begin{aligned} y(0) &= \frac{3}{4}y(-1) - \frac{1}{8}y(-2) + x(0) = 0 \\ y(1) &= \frac{3}{4}y(0) - \frac{1}{8}y(-1) + x(1) = \alpha + \frac{1}{6} \end{aligned}$$

Then C_1 and C_2 should satisfy

$$\begin{aligned} C_1 + C_2 + \frac{\alpha^2}{\alpha^2 - \frac{3}{4}\alpha + \frac{1}{8}} &= y(0) = 0 \\ \frac{1}{2}C_1 + \frac{1}{4}C_2 + \frac{\alpha^3}{\alpha^2 - \frac{3}{4}\alpha + \frac{1}{8}} &= y(1) = \alpha + \frac{1}{6} \end{aligned}$$

solving for C_1 and C_2, we get

$$\begin{aligned} C_1 &= -\frac{\frac{4}{3}\alpha^2 - \frac{1}{12}}{\alpha^2 - \frac{3}{4}\alpha + \frac{1}{8}} = -\frac{4}{3}\frac{\alpha + \frac{1}{4}}{\alpha - \frac{1}{2}} \\ C_2 &= \frac{\frac{1}{3}\alpha^2 - \frac{1}{12}}{\alpha^2 - \frac{3}{4}\alpha + \frac{1}{8}} = \frac{1}{3}\frac{\alpha + \frac{1}{2}}{\alpha - \frac{1}{4}} \end{aligned}$$

It follows that

$$\boxed{y(n) = \frac{1}{\alpha^2 - \frac{3}{4}\alpha + \frac{1}{8}} \left[-\left(\frac{4}{3}\alpha^2 - \frac{1}{12}\right)\left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\alpha^2 - \frac{1}{12}\right)\left(\frac{1}{4}\right)^n + \alpha^{n+2} \right] u(n) , \qquad n \geq 0}$$

3. (10 PTS) Are there choices of α for which at least one of the modes of the system is not excited (i.e., does not appear) at the output? Describe all such $\alpha' s$.

For the mode $\lambda = \frac{1}{2}$ to disappear, we choose α such that , i.e.

$$\alpha = -\frac{1}{4}$$

Similarly, for the mode $\lambda = \frac{1}{4}$ to disappear, we choose α such that $C_2 = 0$, i.e.

$$\alpha = -\frac{1}{2}$$

4. (10 PTS) Find the energy of the output sequence when $\alpha = -1/4$.

$$\begin{aligned} y(n) &= \frac{8}{3}\left[-\frac{1}{16}\left(\frac{1}{4}\right)^n + \left(\frac{-1}{4}\right)^{n+2} \right] u(n) \\ &= \frac{1}{6}\left[\left(\frac{-1}{4}\right)^n - \left(\frac{1}{4}\right)^n \right] u(n) \\ &= \frac{1}{3}\left(\frac{-1}{4}\right)^{2n+1} u(n) \\ &= -\frac{1}{12}\left(\frac{-1}{4}\right)^{2n} u(n) \end{aligned}$$

The energy of the sequence $y(n)$ is given by

$$\begin{aligned}
E_Y &= \sum_{n=-\infty}^{\infty} |y(n)|^2 \\
&= \left(\frac{1}{12}\right)^2 \sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^{4n} \\
&= \frac{1}{144} \sum_{n=0}^{\infty} \left(\frac{1}{256}\right)^{n} \\
&= \frac{1}{144} \cdot \frac{1}{1-\frac{1}{256}} = 0.007
\end{aligned}$$

5. (30 PTS) Find the same complete response as in part 2) above by using the $z-$transform technique.

Only zero state solution can be found by Z-transform.

Zero–state solution

Taking the Z-transform of the difference equation, we find that the Z-transform of the zero–state solution is

$$Y_{zs}(z) - \frac{3}{4}z^{-1}Y_{zs}(z) + \frac{1}{8}z^{-2}Y_{zs}(z) = X(z)$$

$$Y_{zs}(z) = \frac{X(z)}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}} = \frac{z^2X(z)}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)}$$

For $x(n) = \alpha^n u(n)$

$$X(z) = \frac{z}{z-\alpha}\ , \qquad |z| > |\alpha|$$

Then,

$$Y_{zs}(z) = \frac{z^3}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)(z-\alpha)}$$

Define $\bar{Y}(z) = z^{-1}Y_{zs}(z)$. Using partial fractions, $\bar{Y}(z)$ can be written as

$$\begin{aligned}
\bar{Y}(z) &= \frac{z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)(z-\alpha)} \\
&= \frac{A}{z-\frac{1}{4}} + \frac{B}{z-\frac{1}{2}} + \frac{C}{z-\alpha}
\end{aligned}$$

The constants A,B, and C are evaluated as follows

$$\begin{aligned}
A &= \left(z-\frac{1}{4}\right)\bar{Y}(z)|_{z=\frac{1}{4}} = \frac{\frac{1}{4}}{\alpha-\frac{1}{4}} \\
B &= \left(z-\frac{1}{2}\right)\bar{Y}(z)|_{z=\frac{1}{2}} = \frac{-1}{\alpha-\frac{1}{2}} \\
C &= (z-\alpha)\,\bar{Y}(z)|_{z=\alpha} = \frac{\alpha^2}{\alpha^2-\frac{3}{4}\alpha+\frac{1}{8}}
\end{aligned}$$

Substituting A,B, and C, we get

$$\bar{Y}(z) = \frac{\frac{1}{4}}{\alpha-\frac{1}{4}} \cdot \frac{1}{z-\frac{1}{4}} + \frac{-1}{\alpha-\frac{1}{2}} \cdot \frac{1}{z-\frac{1}{2}} + \frac{\alpha^2}{\alpha^2-\frac{3}{4}\alpha+\frac{1}{8}} \cdot \frac{1}{z-\alpha}$$

$$\begin{aligned} Y_{zs}(z) &= \frac{\frac{1}{4}}{\alpha-\frac{1}{4}}\cdot\frac{z}{z-\frac{1}{4}} + \frac{-1}{\alpha-\frac{1}{2}}\cdot\frac{z}{z-\frac{1}{2}} + \frac{\alpha^2}{\alpha^2-\frac{3}{4}\alpha+\frac{1}{8}}\cdot\frac{z}{z-\alpha} \\ &= \frac{1}{\alpha^2-\frac{3}{4}\alpha+\frac{1}{8}}\left[-\left(\alpha-\frac{1}{4}\right)\frac{z}{z-\frac{1}{2}} + \frac{1}{4}\left(\alpha-\frac{1}{2}\right)\frac{z}{z-\frac{1}{4}} + \alpha^2\frac{z}{z-\alpha}\right] \end{aligned}$$

Using inverse Z-transform, $y_{zs}(n)$ is

$$\boxed{y_{zs}(n) = \frac{1}{\alpha^2-\frac{3}{4}\alpha+\frac{1}{8}}\left[-\left(\alpha-\frac{1}{4}\right)\left(\frac{1}{2}\right)^n + \frac{1}{4}\left(\alpha-\frac{1}{2}\right)\left(\frac{1}{4}\right)^n + \alpha^{n+2}\right]u(n)}$$

Zero–input solution

$$y_{zi}(n) = C_1\left(\frac{1}{2}\right)^n + C_2\left(\frac{1}{4}\right)^n$$

Using the initial conditions $y(-1) = -\frac{4}{3}$ and $y(-2) = 0$ to determine C_1 and C_2 we get

$$\begin{aligned} 2C_1 + 4C_2 &= -\frac{4}{3} \\ 4C_1 + 16C_2 &= 0 \end{aligned}$$

By solving these two equations, we find $C_1 = -\frac{4}{3}$ and $C_2 = \frac{1}{3}$. Then,

$$\boxed{y_{zi}(n) = -\frac{4}{3}\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(\frac{1}{4}\right)^n}$$

Complete solution

Adding the zero–input solution to the zero–state solution we obtain the complete solution

$$\begin{aligned} y(n) &= y_{zs}(n) + y_{zi}(n) \\ &= \frac{1}{\alpha^2-\frac{3}{4}\alpha+\frac{1}{8}}\left[-\left(\frac{4}{3}\alpha^2-\frac{1}{12}\right)\left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\alpha^2-\frac{1}{12}\right)\left(\frac{1}{4}\right)^n + \alpha^{n+2}\right]u(n)\,, \qquad n \geq 0 \end{aligned}$$

6. (10 BONUS PTS) Which value of α results in an output sequence with smallest energy?
Let $y(n) = \left[C_1\left(\frac{1}{2}\right)^n + C_2\left(\frac{1}{4}\right)^n + K\alpha^n\right]u(n)$ where

$$\begin{aligned} C_1 &= -\frac{\frac{4}{3}\alpha^2-\frac{1}{12}}{\alpha^2-\frac{3}{4}\alpha+\frac{1}{8}} \\ C_2 &= \frac{\frac{1}{3}\alpha^2-\frac{1}{12}}{\alpha^2-\frac{3}{4}\alpha+\frac{1}{8}} \\ K &= \frac{\alpha^2}{\alpha^2-\frac{3}{4}\alpha+\frac{1}{8}} \end{aligned}$$

Then,

$$\begin{aligned} E_Y &= \sum_{n=-\infty}^{\infty}|y(n)|^2 \\ &= \sum_{n=-\infty}^{\infty} C_1^2\left(\frac{1}{2}\right)^{2n} + C_2^2\left(\frac{1}{4}\right)^{2n} + K^2\alpha^{2n} + 2C_1C_2\left(\frac{1}{8}\right)^n + 2C_1K\left(\frac{\alpha}{2}\right)^n + 2C_2K\left(\frac{\alpha}{4}\right)^n \\ &= \frac{4}{3}C_1^2 + \frac{16}{15}C_2^2 + \frac{K^2}{1-\alpha^2} + \frac{16}{7}C_1C_2 + \frac{2C_1K}{1-\frac{\alpha}{2}} + \frac{2C_2K}{1-\frac{\alpha}{4}} \end{aligned}$$

Then we find the value of α results in an output sequence with smallest energy by setting $\frac{d}{d\alpha}E_Y = 0$ and solving the resulting equation.