EE113: Digital Signal Processing
Prof. A. H. Sayed Oct. 30, 2001

MIDTERM EXAMINATION

(Closed Book)

You are only required to answer THREE problems of your choice. If you answer all problems, you will get additional (bonus) credit.

1. (25 PTS). Consider the sequence

$$x(n) = \left(\frac{1}{3}\right)^n u(n-1) \;+\; \left(\frac{1}{2}\right)^{n-1} u(n-2)$$

   (a) Find its energy.

   (b) Let $y(n) = x(-2n+3)$. For what values of n is $y(n)$ zero?

2. (25 PTS). The response of an LTI system to $x(n) = u(n-2)$ is $y(n) = \left(\frac{1}{2}\right)^{n-2} u(n-4)$.

   (a) Find its impulse response sequence.

   (b) Is this a BIBO stable system? Is it causal? Can you tell what the modes of the system are?

3. (25 PTS). A causal system is described by the difference equation

$$y(n) - y(n-1) + \frac{1}{4}y(n-2) \;=\; x(n-1), \quad y(-1) = 0, \quad y(-2) = -\frac{7}{2}.$$

   (a) Find its zero-input response.

   (b) Find the $z-$transform of the zero-state response of the system when $x(n) = \left(\frac{1}{2}\right)^{n+1} u(n)$.

4. (25 PTS).

   (a) Find the $z-$transform of the sequence

$$x(n) = nu(n-2) \;+\; \left(\frac{1}{2}\right)^{-n} u(n+1)$$

   (b) Evaluate the sum

$$\sum_{n=2}^{\infty} n^2 \left(\frac{1}{3}\right)^n$$

EE113: Digital Signal Processing Fall 2001 Midterm Solutions
Prof. A. H. Sayed Oct. 30, 2001

MIDTERM SOLUTION

(Prepared by Jul Setsu Cha)

1. (a) Since energy of a sequence $x(n)$ is, $\mathcal{E}_x = \sum_{n=-\infty}^{\infty} |x(n)|^2$,

$$
\begin{aligned}
\mathcal{E}_x &= \sum_{n=-\infty}^{\infty} \left| \left(\frac{1}{3}\right)^n u(n-1) + 2\left(\frac{1}{2}\right)^n u(n-2) \right|^2 \\
&= \sum_{n=-\infty}^{\infty} \left(\left(\frac{1}{3}\right)^n u(n-1) + 2\left(\frac{1}{2}\right)^n u(n-2) \right)^2 \\
&= \sum_{n=-\infty}^{\infty} \left(\left(\frac{1}{3}\right)^{2n} u(n-1) + 4\left(\frac{1}{3}\cdot\frac{1}{2}\right)^n u(n-2) + 4\left(\frac{1}{2}\right)^{2n} u(n-2) \right) \\
&= \sum_{n=-\infty}^{\infty} \left(\frac{1}{9}\right)^n u(n-1) + 4\sum_{n=-\infty}^{\infty} \left(\frac{1}{6}\right)^n u(n-2) + 4\sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\right)^n u(n-2) \\
&= \left[\sum_{n=0}^{\infty} \left(\frac{1}{9}\right)^n - \left(\frac{1}{9}\right)^0 \right] + 4\left[\sum_{n=0}^{\infty} \left(\frac{1}{6}\right)^n - \left(\frac{1}{6}\right)^0 - \left(\frac{1}{6}\right)^1 \right] \\
&\quad + 4\left[\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n - \left(\frac{1}{4}\right)^0 - \left(\frac{1}{4}\right)^1 \right] \\
&= \frac{1}{1-\frac{1}{9}} - 1 + 4\left[\frac{1}{1-\frac{1}{6}} - 1 - \frac{1}{6} \right] + 4\left[\frac{1}{1-\frac{1}{4}} - 1 - \frac{1}{4} \right] \\
&= \frac{9}{8} - 1 + 4\left[\frac{1}{30}\right] + 4\left[\frac{1}{12}\right] \\
&= \frac{1}{8} + \frac{2}{15} + \frac{1}{3} \\
&= \frac{71}{120}
\end{aligned}
$$

(b) @$n = 1$, we get $y(1) = x(1)$
@$n = 2$, we get $y(2) = x(-1)$
@$n = 3$, we get $y(3) = x(-3)$
We know that $x(n)$ becomes zero for $n < 1$, and increasing indices of $y(n)$ results in the decrease of the indices of $x(n)$.
Therefore, at $n \geq 2$ or $n > 1$, we get $y(n) = 0$.

2. (a)

$$x(n) = u(n-2) \rightarrow \boxed{S} \rightarrow y(n) = \left(\frac{1}{2}\right)^{n-2} u(n-4)$$

Since the system is TI,

$$x(n+2) = u(n) \quad \rightarrow \boxed{S} \rightarrow \quad y(n+2) = \left(\frac{1}{2}\right)^n u(n-2)$$

And

$$x(n+1) = u(n-1) \quad \rightarrow \boxed{\text{S}} \rightarrow \quad y(n+1) = \left(\frac{1}{2}\right)^{n-1} u(n-3)$$

Then, since the system is linear,

$$\delta(n) = u(n) - u(n-1) = x(n+2) - x(n+1) \rightarrow$$
$$\boxed{\text{S}} \rightarrow y(n+2) - y(n+1) = h(n)$$

Hence,

$$\begin{aligned}
h(n) &= y(n+2) - y(n+1) \\
&= \left(\frac{1}{2}\right)^{n} u(n-2) - \left(\frac{1}{2}\right)^{n-1} u(n-3) \\
&= \left[\left(\frac{1}{2}\right)^{2} \delta(n-2) + \left(\frac{1}{2}\right)^{n} u(n-3)\right] - \left(\frac{1}{2}\right)^{n-1} u(n-3) \\
&= \frac{1}{4}\delta(n-2) + \left[\left(\frac{1}{2}\right)^{n} - \left(\frac{1}{2}\right)^{n-1}\right] u(n-3) \\
&= \frac{1}{4}\delta(n-2) - \left(\frac{1}{2}\right)^{n} u(n-3), \quad \text{Therefore} \\
h(n) &= \frac{1}{4}\delta(n-2) - \left(\frac{1}{2}\right)^{n} u(n-3)
\end{aligned}$$

(b) For LTI systems, BIBO stability requires

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Therefore,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |h(n)| &= \sum_{n=-\infty}^{\infty} \left|\frac{1}{4}\delta(n-2) - \left(\frac{1}{2}\right)^{n} u(n-3)\right| \\
&= \frac{1}{4} + \sum_{n=3}^{\infty} \left(\frac{1}{2}\right)^{n} \\
&= \frac{1}{4} + \left[\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} - \left(\frac{1}{2}\right)^{0} - \left(\frac{1}{2}\right)^{1} - \left(\frac{1}{2}\right)^{2}\right] \\
&= \frac{1}{4} + \left[\frac{1}{1-\frac{1}{2}} - 1 - \frac{1}{2} - \frac{1}{4}\right] \\
&= 2 - 1 - \frac{1}{2} \\
&= \frac{1}{2} < \infty
\end{aligned}$$

Therefore, yes, the system is BIBO stable.
The LTI system is causal iff $h(n) = 0$ for $n < 0$. Since,

$$h(n) = \frac{1}{4}\delta(n-2) - \left(\frac{1}{2}\right)^n u(n-3) = 0, \quad \text{for } n < 2$$

consequently, $h(n) = 0$ for $n < 0$. Therefore this system is causal.
We know that the system has a mode at $\frac{1}{2}$. It could have other modes as well that may not have appeared in $h(n)$.

3. (a) $y_{zi}(n)$:

$$\begin{aligned} \lambda^2 - \lambda + \frac{1}{4} &= 0 \\ \left(\lambda - \frac{1}{2}\right)^2 &= 0 \end{aligned}$$

Hence,

$$y_{zi}(n) = C_1\left(\frac{1}{2}\right)^n + C_2 n\left(\frac{1}{2}\right)^n, \text{ for all n}$$

Using the given initial conditions,

$$\begin{aligned} \text{at } n = -1, \quad 2C_1 - 2C_2 &= 0 \\ \text{at } n = -2, \quad 4C_1 - 8C_2 &= -\frac{7}{2} \end{aligned}$$

Then we get, $C_1 = C_2 = \frac{7}{8}$. Thus,

$$y_{zi}(n) = \frac{7}{8}\left(\frac{1}{2}\right)^n + \frac{7}{8}n\left(\frac{1}{2}\right)^n, \quad \text{for all n}$$

(b) The system becomes unforced for $n \geq 2$. Hence, the impulse response of the relaxed system is

$$h(n) = C_1\left(\frac{1}{2}\right)^n + C_2 n\left(\frac{1}{2}\right)^n, \quad n \geq 2$$

Substituting the relaxed initial conditions into the difference equation we get $h(1) = 1$. Since we know that $h(0) = 0$,

$$\begin{aligned} h(0) = \; 0 &= C_1 \\ h(1) = \; 1 &= C_2\left(\frac{1}{2}\right) \end{aligned}$$

Hence, $C_2 = 2$, and

$$h(n) = 2n\left(\frac{1}{2}\right)^n u(n)$$

Since

$$\begin{aligned} y_{zs}(n) &= h(n) * x(n) \\ Y_{zs}(z) &= H(z)X(z) \end{aligned}$$

Since

$$\begin{aligned} H(z) &= \frac{z}{\left(z-\frac{1}{2}\right)^2}; \quad |z| > \frac{1}{2} \\ X(z) &= \frac{1}{2}\frac{z}{z-\frac{1}{2}}; \quad |z| > \frac{1}{2} \end{aligned}$$

Hence,

$$\begin{aligned} Y_{zs}(z) &= \frac{z}{\left(z-\frac{1}{2}\right)^2}\frac{\frac{1}{2}z}{z-\frac{1}{2}} \\ Y_{zs}(z) &= \frac{\frac{1}{2}z^2}{\left(z-\frac{1}{2}\right)^3}, \quad |z| > \frac{1}{2} \end{aligned}$$

4. (a) For $nu(n-2)$:

Since $nu(n-2) = (n-2)u(n-2) + 2u(n-2)$,

$$\begin{aligned} 2u(n-2) &\leftrightarrow \frac{2}{z(z-1)}, \quad |z| > 1 \\ (n-2)u(n-2) &\leftrightarrow \frac{1}{z(z-1)^2}, \quad |z| > 1 \end{aligned}$$

Then

$$\begin{aligned} nu(n-1) &\leftrightarrow \frac{1}{z(z-1)^2} + \frac{2}{z(z-1)} \\ &= \frac{2z-1}{z(z-1)^2}, \quad |z| > 1 \end{aligned}$$

For $\left(\frac{1}{2}\right)^{-n} u(n+1)$:

$$\begin{aligned} \left(\frac{1}{2}\right)^{-n} u(n+1) &= (2)^n u(n+1) = \frac{1}{2}(2)^{n+1}u(n+1) \\ \frac{1}{2}(2)^{n+1}u(n+1) &\leftrightarrow \frac{1}{2}\frac{z^2}{z-2}, \quad |z| > 2 \end{aligned}$$

Therefore,

$$X(z) = \frac{2z-1}{z(z-1)^2} + \frac{z^2}{2(z-2)}, \quad |z| > 2$$

(b) Let $x(n) = n \cdot n \left(\frac{1}{3}\right)^n u(n)$,

$$\begin{aligned} &\sum_{n=2}^{\infty} n^2 \left(\frac{1}{3}\right)^n \\ &= \left[\sum_{n=\infty}^{\infty} n \cdot n \left(\frac{1}{3}\right)^n u(n) z^{-n}\right]_{z=1} - x(0) - x(1) \end{aligned}$$

$$= \quad [X(z)]_{z=1} - \left(\frac{1}{3}\right)^1$$

Since,

$$n\left(\frac{1}{3}\right)^n u(n) \quad \leftrightarrow \quad -z\frac{d}{dz}\frac{z}{z-\frac{1}{3}} = \frac{1}{3}\frac{z}{\left(z-\frac{1}{3}\right)^2}$$

$$n\left[n\left(\frac{1}{3}\right)^n u(n)\right] \quad \leftrightarrow \quad -\frac{z}{3}\frac{d}{dz}\frac{z}{\left(z-\frac{1}{3}\right)^2}$$

$$= -\frac{z}{3}\frac{\left(z-\frac{1}{3}\right)^2 - 2\left(z-\frac{1}{3}\right)z}{\left((z-\frac{1}{3}\right)^4}$$

Then,

$$X(z=1) \quad = \quad -\frac{1}{3}\frac{\left(\frac{2}{3}\right)^2 - 2\left(\frac{2}{3}\right)}{\left(\frac{2}{3}\right)^4}$$

$$= \quad \frac{3}{2}$$

Therefore,

$$\sum_{n=2}^{\infty} n^2\left(\frac{1}{3}\right)^n = \frac{3}{2} - \frac{1}{3} = \frac{7}{6}$$