EE113: Digital Signal Processing Fall 2014
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SOLUTIONS FOR MIDTERM EXAMINATION

1. (50 PTS) **True** or **False**? Explain or give counter-examples:

   (a) If a system is unstable, then it cannot be FIR.
   Solution: True. It is equivalent to prove that an FIR system is always stable. Suppose we have a bounded input $|x(n)| \leq B$ for all n. The output of an FIR LTI system

   $$y(n) \triangleq \sum_{k=0}^{N} a_k x(n-k)$$

   is bounded since

   $$|y(n)| \leq \sum_{l=0}^{N} |a_k| \cdot |x(n-k)| \leq B \sum_{k=0}^{N} |a_k| \triangleq B' \leq \infty$$

   (b) The system $y(n) = y(n-1) + x^2(n), y(-1) = 0, n \geq 0$ is time-variant.
   Solution: False. The overall system can be regarded as a series cascade of two systems. The first system is $x'(n) = x^2(n), n \geq 0$, which is time-invariant. The second system is $y(n) = y(n-1) + x'(n), y(-1) = 0, n \geq 0$, which is LTI. Therefore, the overall system is time-invariant since the series cascade of time-invariant systems is time-invariant.

   (c) If two systems are relaxed, then their series cascade is also relaxed.
   Solution: True. For the first relaxed system with input $x(n)$ and output $x'(n)$, we have that for any n_0,

   $$\text{if } \; x(n) = 0 \; \text{ for } \; n < n_0 \; \text{ then } \; x'(n) = 0 \; \text{ for } \; n < n_0$$

   The sequence $x'(n)$ is then fed into the second relaxed system with output $y(n)$ so that for any n_0

   $$\text{if } \; x'(n) = 0 \; \text{ for } \; n < n_0 \; \text{ then } \; y(n) = 0 \; \text{ for } \; n < n_0$$

   Combining the above results, we can conclude that for any n_0

   $$\text{if } \; x(n) = 0 \; \text{ for } \; n < n_0 \; \text{ then } \; y(n) = 0 \; \text{ for } \; n < n_0$$

   which is a relaxed system.

   (d) If an LTI system is stable, then its impulse response sequence is an energy sequence.
   Solution: True. For a stable LTI system, the impulse response $h(n)$ is absolutely summable, i.e., we have

   $$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \quad \Rightarrow \quad \left(\sum_{n=-\infty}^{\infty} |h(n)| \right)^2 < \infty$$

Since $|h(n)|$ is nonnegative for any n, we have

$$\sum_{n=-\infty}^{\infty} |h(n)|^2 \leq \left(\sum_{n=-\infty}^{\infty} |h(n)| \right)^2 < \infty$$

and, therefore, $h(n)$ is an energy sequence.

(e) Every periodic sequence is a power sequence.

Solution: False. If the samples of periodic sequences have infinite values, then the statement is not true. For example, the sequence $x(n) = \tan(n\pi/2)$ is periodic but not a power sequence. On the other hand, if the samples are all finite, we can prove that the statement is true. Consider a periodic sequence $x(n)$ with period N_x. Let us denote the energy within one period of $x(n)$ as

$$E_x = \sum_{k=1}^{N_x} |x(k + mN_x)|^2$$

for any integer m. Then, the average power of $x(n)$ can be expressed as

$$\begin{aligned} P_x &= \lim_{N\to\infty} \left(\frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \right) \\ &= \lim_{k\to\infty} \left(\frac{1}{2kN_x+1} \sum_{n=-kN_x}^{kN_x} |x(n)|^2 \right) \\ &= \lim_{k\to\infty} \frac{2kE_x + |x(0)^2|}{2kN_x+1} \\ &= \frac{E_x}{N_x} \end{aligned}$$

where we define $N = kN_x$ and let k go to infinity. Therefore, $x(n)$ is a power sequence.

(f) There cannot exist a sequence that has both infinite power and infinite energy.

Solution: False. Consider the sequence $x(n) = \alpha^n u(n), \alpha > 1$ which has both infinite power and infinite energy.

(g) The convolution of two causal sequences can be a non-causal sequence.

Solution: False. The ROC of a causal sequence is the exterior of a disc. Let us consider two causal sequences $x_1(n)$ and $x_2(n)$ with the ROC $= \{|z| > r_1\}$ and the ROC $= \{|z| > r_2\}$, respectively. The convolution $y(n) = x_1(n) \star x_2(n)$ has the ROC $= \{|z| > \max(r_1, r_2)\}$, which implies that $y(n)$ is also a causal sequence. Another way to see this is to use the definition of the convolution. Suppose that we can write $x_1(n) = x_1'(n)u(n - m_1)$ and $x_2(n) = x_2'(n)u(n - m_2)$ for some sequences $x_1'(n)$ and $x_2'(n)$ and some integers m_1 and m_2. Then, the convolution $y(n) = x_1(n) \star x_2(n)$ is given by

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k) \\ &= \sum_{k=m_1}^{n-m_2} x_1'(k)x_2'(n-k)u(k-m_1)u(n-k-m_2) \end{aligned}$$

$$= \left(\sum_{k=m_1}^{n-m_2} x_1'(k) x_2'(n-k) \right) u(n - m_1 - m_2)$$

which is a causal sequence.

(h) The complete solution of an LTI difference equation coincides with its zero-state solution.
Solution: True. The complete solution can be written as $y(n) = y_{zi}(n) + y_{zs}(n)$. An LTI difference equation implies that the initial conditions are zero. By setting the input $x(n) = 0$, we get $y_{zi}(n) = 0$ and thus $y(n) = y_{zs}(n)$.

(i) The steady-state response of a stable LTI system is the zero sequence.
Solution: False. Consider an stable LTI system with $h(n) = \delta(n)$. When the input sequence is $x(n) = \cos(w_o n)u(n)$, the steady-state response of the system is $y_{zs}(n) = \cos(w_o n)$ which is not a zero sequence.

(j) The zero-input solution of a relaxed system is the zero sequence.
Solution: True. From the definition of a relaxed system, we have that for any n_0,

$$\text{if} \;\; x(n) = 0 \;\; \text{for} \;\; n < n_0 \;\; \text{then} \;\; y(n) = 0 \;\; \text{for} \;\; n < n_0$$

For the input $x(n) = 0$, we always get the output $y(n) = 0$.

2. (15 PTS) A class has N students numbered 1 through N. Let $x(n)$ denote the grade of student n in the midterm examination on a scale of 100. Let $y(n)$ denote the maximum grade of the first n students (i.e., of students $1, 2, ..., n$).

(a) Find a description for the system that relates $y(n)$ to $y(n-1)$ and $x(n)$ for $n \geq 1$.
Solution:

$$\begin{aligned} y(n) &= \max_{1 \leq k \leq n} x(k) \\ &= \max \left\{ \max_{1 \leq k \leq n-1} x(k), x(n) \right\} \\ &= \max \left\{ y(n-1), x(n) \right\} \end{aligned}$$

(b) Is the system causal? linear? stable?
Solution: The system is causal because $y(n)$ does not depend on future values of $x(m)$ for any $m > n$.
The system is non-linear. To see this, let us consider two different input sequences $x_1(n)$ and $x_2(n)$, and denote their corresponding output sequences by

$$y_1(n) = \max_{1 \leq k \leq n} x_1(k), \qquad y_2(n) = \max_{1 \leq k \leq n} x_2(k)$$

If we have the input sequence $x(n) = \alpha x_1(n) + \beta x_2(n)$ for any α and β, we have the following result in general:

$$\begin{aligned} y(n) &= \max_{1 \leq k \leq n} x(k) \\ &= \max_{1 \leq k \leq n} \alpha x_1(k) + \beta x_2(k) \\ &\neq \alpha \max_{1 \leq k \leq n} x_1(k) + \beta \max_{1 \leq k \leq n} x_2(k) \\ &= \alpha y_1(n) + \beta y_2(n) \end{aligned}$$

The system is stable because with any bounded input sequence $|x(n)| \leq 100$ for all $n \geq 1$, we always get a bounded output sequence $|y(n)| \leq 100$.

(c) Assume $x(n) = 50$ for all students. What is the response of the system?

Solution: From the result in (a) we have $y(n) = 50$ for all $n \geq 1$.

3. (25 PTS) A causal system is described by the following second-order difference equation:

$$y(n) + \frac{1}{12}y(n-1) - \frac{1}{12}y(n-2) = x^2(n), \quad y(-1) = 0, \quad y(-2) = -12, \quad n \geq 0$$

where $x(n)$ denotes causal input sequences.

(a) Is the system relaxed? Why or why not?

Solution: No, the system is not relaxed because the initial conditions are not zero.

(b) Is the system linear? Why or why not?

Solution: No, the system is non-linear because the initial conditions are not zero.

(c) Determine the resulting sequence $y(n)$ when $x(n) = \left(\frac{1}{2}\right)^n u(n)$.

Solution: If we introduce a new input sequence

$$x'(n) \triangleq x^2(n) = \left(\frac{1}{4}\right)^n u(n)$$

it is then equivalent to consider the following first-order difference equation:

$$y(n) + \frac{1}{12}y(n-1) - \frac{1}{12}y(n-2) = x'(n), \quad y(-1) = 0, \quad y(-2) = -12, \quad n \geq 0$$

In this system, the output sequence $y(n)$ can be expressed $y(n) = y_{zi}(n) + y_{zs}(n)$. The characteristic function for the homogeneous equation is

$$\lambda^2 + \frac{1}{12}\lambda - \frac{1}{12} = 0$$

Solving it we get the modes of the difference equation:

$$\lambda_1 = -\frac{1}{3}, \;\; \lambda_2 = \frac{1}{4}$$

Therefore, the homogeneous solution can be expressed in the following form:

$$y_{zi}(n) = C_1\left(-\frac{1}{3}\right)^n + C_2\left(\frac{1}{4}\right)^n, \quad n \geq 0$$

Using the initial conditions $y(-1) = 0$ and $y(-2) = -12$, we get

$$\begin{cases} y(-1) = -3C_1 + 4C_2 = 0 \\ y(-2) = 9C_1 + 16C_2 = -12 \end{cases} \implies \begin{cases} C_1 = -\frac{4}{7} \\ C_2 = -\frac{3}{7} \end{cases}$$

The zero-input response is thus

$$y_{zi}(n) = \left[-\frac{4}{7}\left(-\frac{1}{3}\right)^n - \frac{3}{7}\left(\frac{1}{4}\right)^n\right] u(n)$$

We now proceed to find the impulse response $h(n)$ for the following causal LTI system:

$$y(n) + \frac{1}{12}y(n-1) - \frac{1}{12}y(n-2) = x'(n), \qquad \text{relaxed}$$

Setting the input $x(n) = \delta(n)$ we get

$$h(n) + \frac{1}{12}h(n-1) - \frac{1}{12}h(n-2) = \delta(n)$$

Since we know that $h(n) = 0$ for $n < 0$, we can get two conditions $h(-1) = 0$ and $h(0) = 1$ to solve the homogeneous solution:

$$\begin{cases} h(-1) = -3C_1 + 4C_2 = 0 \\ h(0) = C_1 + C_2 = 1 \end{cases} \implies \begin{cases} C_1 = \frac{4}{7} \\ C_2 = \frac{3}{7} \end{cases}$$

Therefore, the impulse response is

$$h(n) = \left[\frac{4}{7}\left(-\frac{1}{3}\right)^n + \frac{3}{7}\left(\frac{1}{4}\right)^n\right]u(n)$$

The zero-state response for $x'(n)$ can be found by

$$\begin{aligned}
y_{zs}(n) &= h(n) \star x'(n) \\
&= \sum_{k=-\infty}^{\infty} \left[\frac{4}{7}\left(-\frac{1}{3}\right)^k + \frac{3}{7}\left(\frac{1}{4}\right)^k\right]u(k)\left(\frac{1}{4}\right)^{n-k} u(n-k) \\
&= \sum_{k=0}^{n} \left[\frac{4}{7}\left(-\frac{1}{3}\right)^k + \frac{3}{7}\left(\frac{1}{4}\right)^k\right]\left(\frac{1}{4}\right)^{n-k} u(n) \\
&= \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} \left[\frac{4}{7}\left(-\frac{1}{3}\right)^k + \frac{3}{7}\left(\frac{1}{4}\right)^k\right] 4^k u(n) \\
&= \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} \left[\frac{4}{7}\left(-\frac{4}{3}\right)^k + \frac{3}{7}\right] u(n) \\
&= \frac{3}{7}(n+1)\left(\frac{1}{4}\right)^n u(n) + \frac{4}{7}\left(\frac{1}{4}\right)^n \frac{1-\left(-\frac{4}{3}\right)^{n+1}}{1+\frac{4}{3}} u(n) \\
&= \frac{3}{7}(n+1)\left(\frac{1}{4}\right)^n u(n) + \frac{12}{49}\left(\frac{1}{4}\right)^n u(n) + \frac{16}{49}\left(-\frac{1}{3}\right)^n u(n) \\
&= \frac{3}{7}n\left(\frac{1}{4}\right)^n u(n) + \frac{33}{49}\left(\frac{1}{4}\right)^n u(n) + \frac{16}{49}\left(-\frac{1}{3}\right)^n u(n)
\end{aligned}$$

The complete solution is then obtained as

$$y(n) = y_{zi}(n) + y_{zs}(n) = \frac{3}{7}n\left(\frac{1}{4}\right)^n u(n) + \frac{12}{49}\left(\frac{1}{4}\right)^n u(n) - \frac{12}{49}\left(-\frac{1}{3}\right)^n u(n)$$

(d) Determine the z-transform and ROC of $y(n)$ from part (c).
Solution: The z-transform of $y(n)$ is

$$Y(z) = \frac{3}{28}\frac{z}{\left(z-\frac{1}{4}\right)^2} + \frac{12}{49}\frac{z}{z-\frac{1}{4}} - \frac{12}{49}\frac{z}{z+\frac{1}{3}}$$

with the ROC as

$$\text{ROC:} \quad \left\{|z| > \frac{1}{3}\right\} \bigcap \left\{|z| > \frac{1}{4}\right\} = \left\{|z| > \frac{1}{3}\right\}$$

(e) What would the response $y(n)$ be if the initial conditions are each multiplied by 14 and the input sequence of part (c) is multiplied by 7?

Solution: Note that scaling $x(n)$ by 7 results in scaling $x'(n) = x^2(n)$ by 49. By the superposition properties of $y_{zi}(n)$ and $y_{zs}(n)$, the response $y(n)$ becomes

$$y_n = 14y_{zi}(n) + 49y_{zs}(n) = 21n\left(\frac{1}{4}\right)^n u(n) + 27\left(\frac{1}{4}\right)^n u(n) + 8\left(-\frac{1}{3}\right)^n u(n)$$

4. (10 PTS) Consider the causal LTI system

$$y(n) = \frac{1}{2}y(n-1) + \left(\frac{1}{3}\right)^{2n} \cos(\pi n)\, u(n-2), \quad n \geq 0$$

Use the z-transform technique to determine $y(n)$.

Solution: We can rewrite the causal LTI system as the following relaxed difference equation:

$$y(n) = \frac{1}{2}y(n-1) + x(n), \quad n \geq 0$$

with the input sequence $x(n)$ as

$$\begin{aligned} x(n) &= \left(\frac{1}{3}\right)^{2n} \cos(\pi n)\, u(n-2) \\ &= \left(\frac{1}{9}\right)^{n} (-1)^n\, u(n-2) \\ &= \left(-\frac{1}{9}\right)^{n} u(n-2) \\ &= \left(\frac{1}{9}\right)^{2} \left(-\frac{1}{9}\right)^{n-2} u(n-2) \end{aligned}$$

The z-transform of $x(n)$ is

$$X(z) = \left(\frac{1}{9}\right)^2 z^{-2} \frac{z}{z+\frac{1}{9}} = \left(\frac{1}{9}\right)^2 \frac{z^{-1}}{z+\frac{1}{9}}, \quad |z| > \frac{1}{9}$$

By taking the z-transform on both sides of the difference equation we get

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z)$$

Then, we have

$$\begin{aligned} Y(z) &= \frac{X(z)}{1-\frac{1}{2}z^{-1}} \\ &= \frac{zX(z)}{z-\frac{1}{2}} \\ &= \left(\frac{1}{9}\right)^2 \frac{1}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{9}\right)} \end{aligned}$$

Note that since the input $x(n)$ is a causal sequence, the output $y(n)$ is also a causal sequence. Therefore, the ROC of $Y(z)$ is $|z| > \frac{1}{2}$. To find $y(n)$, we note that

$$\begin{aligned} Y(z) &= \left(\frac{1}{9}\right)^2 \frac{1}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{9}\right)} \\ &= \left(\frac{1}{9}\right)^2 \left(\frac{\frac{18}{11}}{z-\frac{1}{2}} - \frac{\frac{18}{11}}{z+\frac{1}{9}}\right) \\ &= \frac{2}{99}\left(\frac{1}{z-\frac{1}{2}} - \frac{1}{z+\frac{1}{9}}\right) \end{aligned}$$

The output sequence $y(n)$ is then given by

$$y(n) = \frac{2}{99}\left[\left(\frac{1}{2}\right)^{n-1} - \left(-\frac{1}{9}\right)^{n-1}\right] u(n-1)$$