UCLA
Dept. of Electrical Engineering
EE113: Digital Signal Processing
2002 Final Exam

The solutions are in slanted type.

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1. (**20 points**) Consider the causal system:

$$y(n) = -a^2 y(n-2) + x(n) + bx(n-1),$$

where a and b are **positive real** numbers to be determined.

From the CCDE, we have that

$$Y(z)(1 + a^2 z^{-2}) = X(z)(1 + bz^{-1}) \Rightarrow H(z) = \frac{1 + bz^{-1}}{1 + a^2 z^{-2}} = \frac{z(z+b)}{z^2 + a^2},$$

$$H(e^{j\omega}) = \frac{1 + be^{-j\omega}}{1 + a^2 e^{-2j\omega}}.$$

The zeros are at $z = 0$, $z = -b$; *the poles are at* $z = \pm ja$.

(a) For what range of values of a and b is the system stable and minimum phase?

Poles and zeros have to be inside the unit circle $\Rightarrow a < 1$, $b < 1$.

(b) From the range of values determined in **(a)**, find the values of a and b such that (1) the system does not alter constant signals and (2) the amplitude response of the system at $\omega = \pi/2$ is equal to $\sqrt{2.5}$.

Constant, i.e., zero-frequency, signals go through unaltered $\Rightarrow H(e^{j0}) = 1$,

$$\frac{1+b}{1+a^2} = 1, \quad \Rightarrow \quad b = a^2.$$

$$|H(e^{j\pi/2})| = \left|\frac{1 - jb}{1 - a^2}\right| = \frac{\sqrt{1+b^2}}{1-a^2} = \frac{\sqrt{1+b^2}}{1-b} = \frac{5}{2},$$

$$2(1+b^2) = 5(1 + b^2 - 2b) \;\Rightarrow\; 3b^2 - 10b + 3 = 0 \;\Rightarrow\; b = 1/3.$$

Whence $a = 1/\sqrt{3}$, $b = 1/3$.

(c) For the particular choice of a and b as determined in **(b)**, find the response of the system to the signal $x(n) = (-1/3)^n u(n)$, with the initial conditions $y(-1) = 0$, $y(-2) = 1$.

$$\begin{aligned}
X(z) &= \frac{1}{1+\frac{1}{3}z^{-1}} \\
Y^+(z) &= -\frac{1}{3}z^{-2}(z^2+0+Y^+(z)) + \frac{1+\frac{1}{3}z^{-1}}{1+\frac{1}{3}z^{-1}}, \\
Y^+(z)(1+\frac{1}{3}z^{-2}) &= 1-\frac{1}{3}=\frac{2}{3} \\
Y^+(z) &= \frac{2}{\sqrt{3}}\;\frac{\frac{1}{\sqrt{3}}}{1+\frac{1}{3}z^{-2}}, \\
y(n) &= \frac{2}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)^n \cos(n\pi/2)u(n).
\end{aligned}$$

Remember the time-delay property of the unilateral z-transform:

$$\mathcal{Z}^+\{x(n-k)\} = z^{-k}\left[\sum_{m=1}^{k} x(-m)z^m + X^+(z)\right].$$

2. (**20 points**) Given the 4-point sequence $x(n) = \{\underline{2}, 0, 1, 0\}$, let $X(k)$ be its DFT.

(a) Compute $X(k)$.

From the definition, $X(k) = \sum_{n=0}^{4-1} x(n)e^{-j2\pi kn/4} = \sum_{n=0}^{3} x(n)e^{-j\pi kn/2}$, *we have that*

$$X(k) = 2 + e^{-j\pi n} = \{\underline{3}, 1, 3, 1\}.$$

(b) Compute the IDFT of $Y(k) = (-1)^k X(k)$; compare it to $x(n)$.

$Y(k) = \{\underline{3}, -1, 3, -1\}$. *From the definition,* $y(n) = \frac{1}{4}\sum_{n=0}^{4-1} Y(k)e^{j2\pi kn/4} = \frac{1}{4}\sum_{n=0}^{3} Y(k)e^{j\pi kn/2}$, *we have that*

$$y(n) = \frac{1}{4}(3 - e^{j\pi n/2} + 3e^{j\pi n} - e^{j3\pi n/2}) = \{\underline{1}, 0, 2, 0\}.$$

We see that $y(n) = x((n+2)\text{mod } 4)$.

3. (**20 points**) Consider the sequences $x_1(n) = \{\underline{1}, 0, 1\}$ and $x_2(n) = \{\underline{-1}, 1\}$. Let $y(n) = x_1(n) * x_2(n)$ be their linear convolution.

(a) Compute $y(n)$ by using the z-transform.

From the definition, $X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$, *we have that*

$$\begin{aligned}
X_1(z) &= 1 + z^{-2}, \\
X_2(z) &= -1 + z^{-1}, \\
Y(z) = X_1(z)X_2(z) &= -1 + z^{-1} - z^{-2} + z^{-3} \;\Rightarrow\; y(n) = \{\underline{-1}, 1, -1, 1\}.
\end{aligned}$$

(b) Compute $y(n)$ by using an N-point DFT. (Use the smallest N that guarantees the correct result! Justify your choice.)

The smallest N is $N = N_1 + N_2 = 4$, where $N_1 = 3$ and $N_2 = 2$ are the lengths of $x_1(n)$ and $x_2(n)$ respectively. From the definition,

$$\begin{aligned} X_1(k) &= 1 + e^{-j\pi k} = \{\underline{2}, 0, 2, 0\}, \\ X_2(k) &= -1 + e^{-jk\pi/2} = \{\underline{0}, -1 - j, -2, -1 + j\}, \\ Y(k) = X_1(k)X_2(k) &= \{\underline{0}, 0, -4, 0\}. \end{aligned}$$

From the definition of IDFT, $y(n) = \frac{1}{4}(-4)e^{j\pi n} = \{\underline{-1}, 1, -1, 1\}$.

4. (**20 points**) Consider the continuous-time signal $x_a(t) = 15 + 25\cos(6000\pi t)$. The signal is sampled at a frequency $f_s = 12$ kHz ($\Omega_s = 24,000\pi$ rad/s). The resulting discrete-time sequence is processed by the relaxed and causal LTI system described by

$$y(n) = 0.25y(n-2) + x(n) + 0.8x(n-2).$$

(a) Draw the pole-zero diagram of the LTI system.

We have that

$$Y(z)(1 - 0.25z^{-2}) = X(z)(1 + 0.8z^{-2}) \Rightarrow H(z) = \frac{1 + 0.8z^{-2}}{1 - 0.25z^{-2}}.$$

From the above we see that there are two zeros at $z = \pm j\sqrt{0.8}$ and two poles at $z = \pm 0.5$.

(b) Is the system of **(a)** stable? Is it minimum-phase?

It is stable because the ROC includes the unit circle (the system is causal). It is minimum phase because all zeros are inside the unit circle.

(c) What is the transfer function of the LTI system in the frequency domain, $H(e^{j\omega})$?

From (a) we have that

$$H(e^{j\omega}) = \frac{1 + 0.8e^{-j2\omega}}{1 - 0.25e^{-j2\omega}}.$$

(d) Compute the expression for the discrete-time sequence $y(n)$ at the output of the LTI system.

The input to the linear system is $x(n) = x_a(nT_s)$, where $T_s = 1/f_s = 1/12000$,

$$x(n) = 15 + 25\cos\left(6000\pi\frac{n}{12000}\right) = 15 + 25\cos(n\pi/2) = 15\cos(\omega_0 n) + 25\cos(\omega_1 n),$$

i.e, $x(n)$ is the superposition of two frequency components, one at $\omega_0 = 0$, and one at $\omega_1 = \pi/2$. Therefore,

$$y(n) = 15|H(e^{j\omega_0})|\cos(\omega_0 n + \angle H(e^{j\omega_0})) + 25|H(e^{j\omega_1})|\cos(\omega_1 n + \angle H(e^{j\omega_1})).$$

$$\begin{aligned} H(e^{j\omega_0}) &= H(e^{j0}) = \frac{1+0.8}{1-0.25} = \frac{36}{15}, \\ H(e^{j\omega_1}) &= H(e^{j\pi/2}) = \frac{1-0.8}{1+0.25} = \frac{4}{25}. \end{aligned}$$

Therefore

$$y(n) = 36 + 4\cos(n\pi/2).$$

5. (**20 points**) Consider the bandpass real signal $x_a(t)$, whose spectrum is non-zero only between frequencies between 6 kHz and 8 kHz (see Fig. 1). The signal is sampled at frequency $f_s = 6$ kHz (assume the sampling is ideal), and let $x_s(t)$ be the sampled signal.

   (a) Draw the magnitude spectrum of $x_s(t)$, $|X_s(\Omega)|$.

   The magnitude spectrum on $X_s(\Omega)$ is obtained by replicating $|X_a(\Omega)|$ an infinite number of times, with periodicity $\Omega_s = 12000\pi$ rad/s, see Fig. 2.

   (b) Can $x_a(t)$ be recovered from $x_s(t)$ by a filtering operation? If so, what kind of filter is required?

   The signal $x_a(t)$ can be recovered from $x_s(t)$ by a bandpass filtering operation. The magnitude spectrum of the filter, $|H(\Omega)|$, is shown in Fig. 2 with a dashed line.

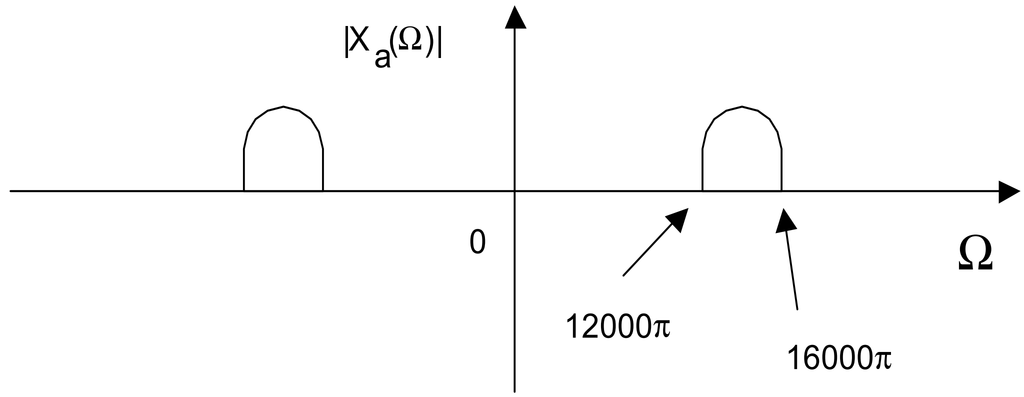

Fig. 1

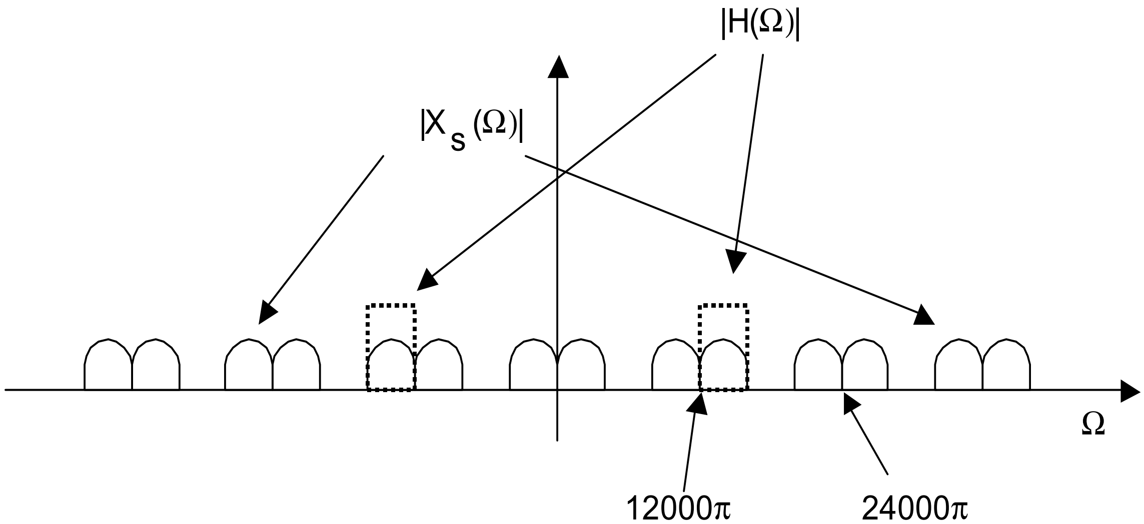

Fig. 2