FINAL EXAMINATION (Open Book)

1. (**20 PTS**) **True** or **False**? Explain or give counter-examples:

   (a) The DTFT of the impulse response of a stable LTI system can exhibit Gibbs phenomenon.
   Solution:
   False. The impulse response of a stable LTI system $h(n)$ is absolutely summable. In this case, $H_N(e^{j\omega}) = \sum_{n=-N}^{N} h(n)e^{-j\omega n}$ uniformly converges towards $H(e^{j\omega})$ for every ω as N goes to infinity, this implies that $H_N(e^{j\omega})$ pointwise converges towards $H(e^{j\omega})$ for every ω so that discrepancies between the two functions will not occur.

   (b) If $x(n)$ is real and even then so is its DTFT.
   Solution:
   True. If $x(n)$ is real, then $x(n) = x^*(n)$. This implies that $X(e^{j\omega}) = X^*(e^{-j\omega})$. Moreover if $x(n)$ is even, then $x(n) = x(-n)$. This implies that $X(e^{j\omega}) = X(e^{-j\omega})$. Thus, when $x(n)$ is real and even, then $X(e^{j\omega}) \overset{(1)}{=} X(e^{-j\omega}) \overset{(2)}{=} X^*(e^{-j\omega})$. The first equality implies that $X(e^{j\omega})$ is even, the second equality implies that $X(e^{-j\omega})$ is real, which also means that $X(e^{j\omega})$ is real because of the first equality. Therefore $X(e^{j\omega})$ is real and even.

   (c) The circular convolution of two sequences can be determined by computing the linear convolution of their periodic embedding.
   Solution:
   True. Let $x(n)$ and $y(n)$ denote two finite sequences defined over $n = 0, 1, \cdots, N-1$. Let $x_p(n)$ and $y_p(n)$ denote their periodic versions, i.e.

$$x_p(n) = \sum_{m=-\infty}^{\infty} x(n+mN), \qquad y_p(n) = \sum_{m=-\infty}^{\infty} x(n+mN)$$

   Let $z(n) = x(n) \circ y(n)$ and $z_p(n) = x_p(n) \star y_p(n)$, then we can show that

$$z_p(n) = \sum_{m=-\infty}^{\infty} z(n+mN)$$

   i.e., if we compute the linear convolution of $x_p(n)$ and $y_p(n)$, the solution will be the periodic version of the sequence $z(n)$ obtained by doing the circular convolution of $x(n)$ and $y(n)$. This can be shown as follows. The circular convolution of $x(n)$ and $y(n)$ is given by:

$$z(n) = \sum_{k=0}^{N-1} x(k)y((n-k) \mod N)$$

   Note that

$$x(k) = x_p(k) \qquad \text{and} \qquad y((n-k) \mod N) = y_p(n-k)$$

thus, the circular convolution of $x(n)$ and $y(n)$ can be written as:

$$z(n) = \sum_{k=0}^{N-1} x_p(k) y_p(n-k)$$

where $n = 0, 1, \cdots, N-1$.
The linear convolution of $x_p(n)$ and $y_p(n)$ is given by:

$$\begin{aligned} z_p(n) &= \sum_{k=\infty}^{\infty} x_p(k) y_p(n-k) \\ &= \sum_{m=-\infty}^{\infty} \sum_{l=0}^{N-1} x_p(l+mN) y_p(n-(l+mN)) \\ &= \sum_{m=-\infty}^{\infty} \sum_{l=0}^{N-1} x_p(l) y_p((n-mN)-l) \\ &= \sum_{m=-\infty}^{\infty} z(n-mN) \\ &= \sum_{m=-\infty}^{\infty} z(n+mN) \end{aligned}$$

(d) If $x(n)$ is causal with samples in the interval $n_o \leq n \leq n_o + N - 1$, then $x(n)$ can be recovered from its $N-$point DFT for any n_o, given that the structure of $x(n)$ is known.
Solution:
True. Let $X(k)$ denote the N-point DFT of $x(n)$. If $n_o = 0$, then $x(n) = \text{IDFT}(X(k))$. If $n_o > 0$, then $x_p(n) = \text{IDFT}(X(k))$, where $x_p(n)$ represents the periodic version of $x(n)$, i.e. $x_p(n) = \sum_{m=-\infty}^{\infty} x(n+Nm)$. Given that we know the format of $x(n)$, we can recover $x(n)$ from $x_p(n)$ as follows:

$$x(n) = x_p(n) \qquad \text{for } n = n_o, \cdots, N-1$$

and

$$x(n) = x_p(N-n) \qquad \text{for } n = N, \cdots, n_o + N - 1$$

(e) Let $x(n) = \cos(\frac{2\pi}{3}n)$ be a sequence defined over $0 \leq n \leq 5$. The $6-$point DFT $X(k)$ of $x(n)$ has a single nonzero coefficient at location $k_o = 2$ over the interval $0 \leq k \leq 5$.
Solution:
False. We note first that $x(n) = \cos(\frac{2\pi}{3}n) = \cos(\frac{2\pi}{6}2n)$. Then

$$X(k) = 3\left(\delta(k-2) + \delta(k-(6-2))\right) = 3\left(\delta(k-2) + \delta(k-4)\right)$$

$X(k)$ has therefore two nonzero coefficients, one at $k = 2$ and another one at $k = 4$.

(f) If $X(e^{j\omega}) = \cos^2(\omega)$, then the energy of the sequence $x(-n+2)$ is 3/8.
Solution:
True. $X(e^{j\omega}) = \cos^2(\omega) = \frac{1}{2}\left(1 + \cos(2\omega)\right) = \frac{1}{2}\left(1 + \frac{e^{j2\omega} + e^{-j2\omega}}{2}\right)$, then

$$x(n) = \frac{1}{2}\delta(n) + \frac{1}{4}\delta(n+2) + \frac{1}{4}\delta(n-2)$$

The energy of $x(n)$ is therefore $\frac{1}{4} + \frac{1}{16} + \frac{1}{16} = \frac{3}{8}$. The energy of $x(-n+2)$ is equal to the energy of $x(n)$. Thus, the energy of $x(-n+2)$ is $\frac{3}{8}$.

(g) The magnitude gain of a stable LTI filter $H(z) = z^{-512}/(1 - 0.5z^{-1})$ at the frequency $f = F_s/4$ (one-fourth of the sampling frequency F_s) is $2/\sqrt{5}$.
Solution:
True. Since the filter is a stable LTI system, its frequency response is given by:

$$H(e^{j\omega}) = e^{-j512\omega}/(1 - 0.5e^{-j\omega})$$

For $f = F_s/4$, the corresponding normalized frequency is: $\omega = 2\pi\frac{f}{F_s} = \frac{\pi}{2}$, thus the magnitude of the frequency response at $\omega = \frac{\pi}{2}$ is:

$$|H(e^{j\omega})|_{\omega=\frac{\pi}{2}} = 1/|1 - 0.5j| = 2/|2 - j| = 2/\sqrt{5}$$

(h) There are only two sequences in the range $0 \le n \le N-1$ for which $x(n) = x(n) \circ x(n)$.
Solution:
False. Let $X(k)$ denote the N–point DFT of $x(n)$, then we have the following relation:

$$X(k) = X^2(k)$$

which means that at any k in the range $0 \le k \le N-1$, $X(k)$ can take a value of 1 or 0. This leads to 2^N choices of $X(k)$ that satisfies the above relation. For instance, we can choose $X(k)$ to be 1 for all k or we can choose it as follows:

$$X(k) = \begin{cases} 1, & 0 \le k \le L \\ 0, & L+1 \le k \le N-1 \end{cases}$$

or we can choose it as follows

$$X(k) = \begin{cases} 0, & 0 \le k \le L \\ 1, & L+1 \le k \le N-1 \end{cases}$$

(i) There are only two sequences for which $x(n) = x(n) \star x(n)$.
Solution:
False. Let $X(e^{j\omega})$ denote the N–point DFT of $x(n)$, then we have the following relation:

$$X(e^{j\omega}) = X^2(e^{j\omega})$$

which means that at any ω in the range $-\pi \le \omega \le \pi$, $X(e^{j\omega})$ can take a value of 1 or 0. This leads to infinitely many choices of $X(e^{j\omega})$ that satisfies the above relation. For instance, we can choose $X(e^{j\omega})$ to be 1 for all ω or we can choose it as follows:

$$X(e^{j\omega}) = \begin{cases} 1, & -\pi \le X(e^{j\omega}) \le 0 \\ 0, & 0 < X(e^{j\omega}) \le \pi \end{cases}$$

or we can choose it as follows

$$X(e^{j\omega}) = \begin{cases} 0, & -\pi \le X(e^{j\omega}) \le 0 \\ 1, & 0 < X(e^{j\omega}) \le \pi \end{cases}$$

(j) If the Nyquist's rate for $x(t)$ is F_s Hz, then the Nyquist's rate for $\sin(2\pi F_s t)x(3t)$ is $\frac{5}{3}F_s$ Hz.
Solution:
False. Let $X(e^{j\Omega})$ denote the Fourier Transform (FT) of $x(t)$ and let F denotes its highest frequency in Hz. Then the FT of $y(t) = x(3t)$ is given by $Y(j\Omega) = \frac{1}{3}X\left(\frac{j\Omega}{3}\right)$, which implies that the highest frequency in $y(t) = x(3t)$ is $3F$. The FT of

$$z(n) = \sin(2\pi F_s t)x(3t) = \sin(2\pi F_s t)y(t) = \frac{1}{2j}\left(e^{-j2\pi F_s t} + e^{j2\pi F_s t}\right)y(t)$$

is given by

$$Z(j\Omega) = \frac{1}{2j}\left(Y(j\Omega + j2\pi F_s) + Y(j\Omega - j2\pi F_s)\right)$$

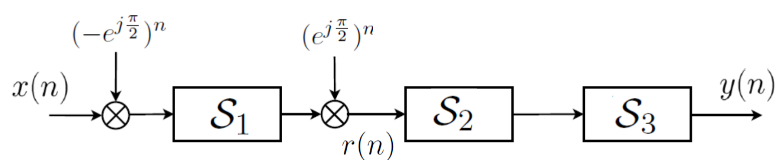
which implies that the highest frequency in $z(t)$ is $3F + F_s$. Since F_s is the Nyquist's rate for $x(t)$, then $F = \frac{F_s}{2}$. Therefore the highest frequency in $z(t)$ is $3F + F_s = 3\frac{F_s}{2} + F_s$, which implies that the Nyquist's rate of $z(n)$ is $2\left(3\frac{F_s}{2} + F_s\right) = 5F_s$.

2. (**40 PTS**) A causal system, with input $x(n)$ and output $y(n)$, consists of the series cascade of several elements in the following succession:

   1. A modulator ($\mathcal{M}_1$), which transforms an input sequence $a(n)$ into $(-e^{j\frac{\pi}{2}})^n a(n)$.
   2. An LTI system ($\mathcal{S}_1$) with impulse response sequence $h_1(n) = u(n) - u(n-2)$.
   3. A second modulator ($\mathcal{M}_1$), which transforms an input sequence $a(n)$ into $(e^{j\frac{\pi}{2}})^n a(n)$.
   4. A second LTI system ($\mathcal{S}_2$) with an impulse response sequence $h_2(n)$ of duration $0 \le n \le 3$ and whose 4−point DFT $H_2(k)$ is given by $H_2(0) = 0$, $H_2(1) = 1$, $H_2(2) = 0$ and $H_2(3) = 1$.
   5. A third LTI system ($\mathcal{S}_3$) with transfer function $H_3(z) = 1/(1 - 0.5z^{-1})$.

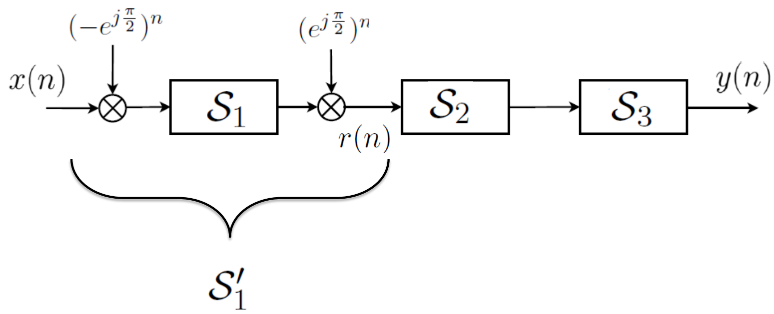
Questions:

(a) Draw a high-level block diagram illustrating the mapping from $x(n)$ to $y(n)$.
Solution:
The block diagram is as follows:



Note that we introduced $r(n)$ which denotes the input to the second system $\mathcal{S}_2$.

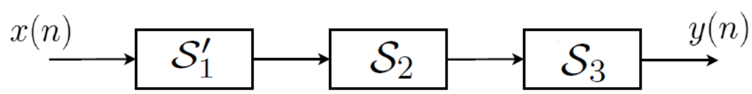
(b) Is the system LTI?
Solution:
Let us first denote by $\mathcal{S}_1'$ the equivalent system that maps $x(n)$ to $r(n)$, as shown in this figure:



We will first show that $\mathcal{S}_1'$ is an LTI system. $h_1(n)$ is the impulse response of $\mathcal{S}_1$. Then $r(n)$ is related to $x(n)$ as follows:

$$\begin{aligned}
r(n) &= e^{j\frac{\pi}{2}n}\left[h_1(n) \star \Big((-e^{j\frac{\pi}{2}})^n x(n)\Big)\right] \\
&= e^{j\frac{\pi}{2}n}\left[\sum_{k=-\infty}^{\infty} h_1(k)(-e^{j\frac{\pi}{2}})^{n-k}x(n-k)\right] \\
&= e^{j\frac{\pi}{2}n}(-e^{j\frac{\pi}{2}})^n\left[\sum_{k=-\infty}^{\infty} h_1(k)(-e^{j\frac{\pi}{2}})^{-k}x(n-k)\right] \\
&= j^n(-j)^n\left[\sum_{k=-\infty}^{\infty} j^k h_1(k)x(n-k)\right] \\
&= \sum_{k=-\infty}^{\infty} j^k h_1(k)x(n-k)
\end{aligned}$$

We notice that the mapping from $x(n)$ to $r(n)$ is linear and time invariant. Therefore the equivalent system $\mathcal{S}'_1$ is linear and time invariant. Moreover, $h_1'(n) \triangleq j^n h_1(n)$ can be seen as the impulse response of $\mathcal{S}_1'$. We can therefore see the overall system as a cascade of three LTI systems: $\mathcal{S}_1'$, $\mathcal{S}_2$ and $\mathcal{S}_3$, as shown in the following figure:



We then conclude that the overall system that maps $x(n)$ to $y(n)$ is LTI.

(c) What is the ROC of $H_3(z)$?
Solution:
Since the overall system is causal, the third system must be causal. This implies that the ROC of $H_3(z)$ is $|z| > \frac{1}{2}$.

(d) What is the frequency response of the overall system?
Solution:
Since the overall system is a cascade of three LTI systems, let us first find the frequency response of each LTI block.

- The first system $\mathcal{S}_1'$ has the following impulse response:

$$h_1'(n) = j^n h(n) = e^{j\frac{\pi n}{2}} h(n)$$

where $h_1(n) = u(n) - u(n-2) = \delta(n) + \delta(n-1)$. The DTFT of $h_1(n)$ over the interval $\omega \in [-\pi, \pi]$ is given by

$$H_1(e^{j\omega}) = 1 + e^{-j\omega}$$

Thus, the DTFT of $h_1'(n)$ over the same interval is given by

$$H_1'(e^{j\omega}) = H_1(e^{j(\omega - \frac{\pi}{2})}) = 1 + e^{-j(\omega - \frac{\pi}{2})} = 1 + je^{-j\omega}$$

– The impulse response of the second system can be obtained by applying the inverse DFT:

$$h_2(n) = \frac{1}{N} \sum_{k=0}^{N-1} H_2(k) e^{j\frac{2\pi}{N}nk}$$

where $N = 4$. Thus,

$$\begin{aligned} h_2(n) &= \frac{1}{4} \sum_{k=0}^{3} H_2(k) e^{j\frac{\pi}{2}nk} \\ &= \frac{1}{4} \left(H_2(1) e^{j\frac{\pi}{2}n} + H_2(3) e^{j\frac{3\pi}{2}n} \right) \\ &= \frac{1}{4} \left(e^{j\frac{\pi}{2}n} + e^{j\frac{3\pi}{2}n} \right) \end{aligned}$$

Therefore, $h_2(0) = \frac{1}{2}$, $h_2(1) = 0$, $h_2(2) = -\frac{1}{2}$ and $h_2(3) = 0$ or equivalently

$$h_2(n) = \frac{1}{2}\delta(n) - \frac{1}{2}\delta(n-2)$$

Thus, the DTFT of $h_2(n)$ over the interval $\omega \in [-\pi, \pi]$ is given by:

$$H_2(e^{j\omega}) = \frac{1}{2} - \frac{1}{2}e^{-j2\omega}$$

– The third system is a stable LTI system since the ROC of $H_3(z)$ includes the unit circle, thus the the DTFT of $h_3(n)$ over the interval $\omega \in [-\pi, \pi]$ is given by:

$$H_3(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

Therefore, the frequency response of the overall system is

$$\begin{aligned} H(e^{j\omega}) &= H_1'\left(e^{j\omega}\right) H_2(e^{j\omega}) H_3(e^{j\omega}) \\ &= \left(1 + je^{-j\omega}\right) \left(\frac{1}{2} - \frac{1}{2}e^{-j2\omega}\right) \frac{1}{1 - \frac{1}{2}e^{-j\omega}} \\ &= \left(\frac{1}{2} + \frac{j}{2}e^{-j\omega} - \frac{1}{2}e^{-j2\omega} - \frac{j}{2}e^{-3j\omega}\right) \frac{1}{1 - \frac{1}{2}e^{-j\omega}} \end{aligned}$$

(e) What is the impulse response sequence of the system?
Solution:
The impulse response $h(n)$ can be deduced from its DTFT $H(e^{j\omega})$:

$$\begin{aligned} h(n) &= \frac{1}{2}\left(\frac{1}{2}\right)^n u(n) + \frac{j}{2}\left(\frac{1}{2}\right)^{n-1} u(n-1) - \frac{1}{2}\left(\frac{1}{2}\right)^{n-2} u(n-2) - \frac{j}{2}\left(\frac{1}{2}\right)^{n-3} u(n-3) \\ &= \frac{1}{2}\left(\frac{1}{2}\right)^n u(n) + j\left(\frac{1}{2}\right)^n u(n-1) - 2\left(\frac{1}{2}\right)^n u(n-2) - 4j\left(\frac{1}{2}\right)^n u(n-3) \end{aligned}$$

$h(n)$ can be equivalently written as:

$$h(n) = \begin{cases} 0, & n < 0 \\ \frac{1}{2}, & n = 0 \\ \frac{1}{4} + \frac{1}{2}j, & n = 1 \\ -\frac{3}{8}j, & n = 2 \\ \left(\frac{1}{2}\right)^n \left(-\frac{3}{2} - 3j\right), & n \geq 3 \end{cases}$$

(f) What is the energy of the impulse response sequence of the system?
Solution:
The energy of the impulse response $h(n)$ is

$$\begin{aligned} \sum_{n=0}^{\infty} |h(n)|^2 &= \frac{1}{4} + \frac{1}{16} + \frac{1}{4} + \frac{9}{64} + \left(\frac{9}{4} + 9\right) \sum_{n=3}^{\infty} \left(\frac{1}{4}\right)^n \\ &= \frac{45}{64} + \left(\frac{45}{4}\right)\left(\frac{1}{4}\right)^3 \frac{1}{1 - \frac{1}{4}} \\ &= \frac{45}{64} + \left(\frac{45}{4}\right)\left(\frac{1}{4}\right)^3 \frac{4}{3} \\ &= \frac{45}{64} + \frac{15}{64} = \frac{15}{16} \end{aligned}$$

(g) Determine a difference equation relating $y(n)$ directly to $x(n)$, for arbitrary causal sequences $x(n)$.
Solution:
We can deduce the difference equations of the overall system from its frequency response, which is given by:

$$H(e^{j\omega}) = \left(\frac{1}{2} + \frac{j}{2}e^{-j\omega} - \frac{1}{2}e^{-j2\omega} - \frac{j}{2}e^{-3j\omega}\right) \frac{1}{1 - \frac{1}{2}e^{-j\omega}} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

This implies that,

$$Y(e^{j\omega}) = \frac{1}{2}e^{-j\omega}Y(e^{j\omega}) + \left(\frac{1}{2} + \frac{j}{2}e^{-j\omega} - \frac{1}{2}e^{-j2\omega} - \frac{j}{2}e^{-3j\omega}\right) X(e^{j\omega})$$

Therefore,

$$y(n) = \frac{1}{2}y(n-1) + \frac{1}{2}x(n) + \frac{j}{2}x(n-1) - \frac{1}{2}x(n-2) - \frac{j}{2}x(n-3)$$

(f) If the system is operating at F_s Hz, what is the magnitude gain at $f = F_s/4$?
Solution:
The corresponding normalized frequency is $\omega = 2\pi\frac{f}{F_s} = \frac{\pi}{2}$. The magnitude gain at $\omega = \frac{\pi}{2}$ is given by

$$\begin{aligned}|H(e^{j\omega})|_{\omega=\frac{\pi}{2}} &= \left|\left(\frac{1}{2} + \frac{j}{2}e^{-j\frac{\pi}{2}} - \frac{1}{2}e^{-j2\frac{\pi}{2}} - \frac{j}{2}e^{-3j\frac{\pi}{2}}\right)\frac{1}{1-\frac{1}{2}e^{-j\frac{\pi}{2}}}\right| \\ &= \left|\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)\frac{1}{1+\frac{1}{2}j}\right| = \frac{4}{\sqrt{5}}\end{aligned}$$

3. (**30 PTS**) Consider a real-valued sequence $x(n)$ of length N, where N is even and let $X(k)$ denote its N–point DFT. Define the following sequence

$$y(n) = \{x(N-1), \ldots, x(1), x(0)\}$$

(a) Let $Y(k)$ denote the N–point DFT of $y(n)$. How is $Y(k)$ related to $X(k)$?
Solution:
We note that $y(n)$ is obtained from $x(n)$ through two steps:

1- Circular time reversal: $y_1(n) = x(-n \mod N)$
2- Circular time shift: $y(n) = y_1(n+1) = x[-(n+1) \mod N]$

The first step implies that

$$Y_1(k) = X(-k \mod N)$$

The second step implies that

$$Y(k) = e^{j\frac{2\pi}{N}k}Y_1(k) = e^{j\frac{2\pi}{N}k}X(-k \mod N)$$

(b) We embed $x(n)$ and $y(n)$ into a new sequence $w(n)$ of length $4N$ in the following manner:

$$w(n) = \left\{\underbrace{0,\ldots,0}_{N/2 \text{ zeros}}, x(0), x(1), \ldots, x(N-1), \underbrace{0,0,\ldots,0}_{N \text{ zeros}}, y(0), y(1), \ldots, y(N-1), \underbrace{0,\ldots,0}_{N/2 \text{ zeros}}\right\}$$

We compute two $4N-$point DFTs in succession as follows:

$$w(n) \xrightarrow{\text{DFT}} W(k) \xrightarrow{\text{DFT}} r(n)$$

Observe that the second transformation is a direct DFT and not an inverse DFT. If $R(k)$ denotes the $4N$–point DFT of $r(n)$, how is $R(k)$ related to $X(k)$?
Solution:
Let $W(k)$ denote the $4N$–point DFT of $w(n)$.

– We first know $r(n) = (4N)w(-n \mod 4N)$. Thus,

$$R(k) = (4N)W(-k \mod 4N)$$

– Let us now establish the relation between $W(k)$ and $X(k)$. Using the definition of $4N$–point DFT, the $4N$–point DFT $W(k)$ over $k = 0, 1, \ldots, 4N-1$ is given by:

$$\begin{aligned}
W(k) &= \sum_{n=0}^{4N-1} w(n) e^{-j\frac{2\pi}{4N}nk} \\
&= \sum_{n=\frac{N}{2}}^{\frac{N}{2}+N-1} w(n) e^{-j\frac{2\pi}{4N}nk} + \sum_{n=\frac{N}{2}+2N}^{\frac{N}{2}+2N+N-1} w(n) e^{-j\frac{2\pi}{4N}nk} \\
&= \sum_{n=\frac{N}{2}}^{\frac{N}{2}+N-1} x\left(n - \frac{N}{2}\right) e^{-j\frac{2\pi}{4N}nk} + \sum_{n=\frac{N}{2}+2N}^{\frac{N}{2}+2N+N-1} y\left(n - \frac{N}{2} - 2N\right) e^{-j\frac{2\pi}{4N}nk} \\
&= \sum_{n'=0}^{N-1} x\left(n'\right) e^{-j\frac{2\pi}{4N}\left(n' + \frac{N}{2}\right)k} + \sum_{n''=0}^{N-1} y\left(n''\right) e^{-j\frac{2\pi}{4N}\left(n'' + \frac{N}{2} + 2N\right)k}
\end{aligned}$$

where we introduced the following variable change in the last step

$$n' = n - \frac{N}{2} \qquad \text{and} \qquad n'' = n - \frac{N}{2} - 2N$$

We can equivalently write $W(k)$ as follows:

$$\begin{aligned}
W(k) &= \sum_{n=0}^{N-1} x\left(n\right) e^{-j\frac{2\pi}{4N}\left(n + \frac{N}{2}\right)k} + \sum_{n=0}^{N-1} y\left(n\right) e^{-j\frac{2\pi}{4N}\left(n + \frac{N}{2} + 2N\right)k} \\
&= e^{-j\frac{\pi}{4}k} \sum_{n=0}^{N-1} x\left(n\right) e^{-j\frac{2\pi}{4N}nk} + e^{-j\frac{5\pi}{4}k} \sum_{n=0}^{N-1} y\left(n\right) e^{-j\frac{2\pi}{4N}nk}
\end{aligned}$$

Let $k = 4\ell$ where $\ell = 0, 1, \ldots, N-1$. Then,

$$\begin{aligned}
W(4\ell) &= e^{-j\frac{\pi}{4}4\ell} \sum_{n=0}^{N-1} x\left(n\right) e^{-j\frac{2\pi}{4N}n4\ell} + e^{-j\frac{5\pi}{4}4\ell} \sum_{n=0}^{N-1} y\left(n\right) e^{-j\frac{2\pi}{4N}n4\ell} \\
&= (-1)^{\ell} \sum_{n=0}^{N-1} x\left(n\right) e^{-j\frac{2\pi}{N}n\ell} + (-1)^{l} \sum_{n=0}^{N-1} y\left(n\right) e^{-j\frac{2\pi}{N}n\ell} \\
&= (-1)^{\ell} X(\ell) + (-1)^{\ell} Y(\ell)
\end{aligned}$$

We now use the relation between $Y(k)$ and $X(k)$ already established in part (a). Then,

$$\begin{aligned}
W(4\ell) &= (-1)^{\ell} X(\ell) + (-1)^{\ell} Y(\ell) \\
&= (-1)^{\ell} X(\ell) + (-1)^{\ell} e^{j\frac{2\pi\ell}{N}} X(-\ell \mod N)
\end{aligned}$$

We already know from the previous step that

$$R(k) = 4N \cdot W(-k \mod 4N)$$

for any $k = 0, 1, \ldots, 4N-1$. Thus,

$$\begin{aligned}
R(4\ell) &= 4N \cdot W(-4\ell \mod 4N) \\
&\overset{(a)}{=} 4N\left[(-1)^{-\ell \mod N} X(-\ell \mod N) + (-1)^{-\ell \mod N} e^{j\frac{2\pi(-\ell \mod N)}{N}} X(\ell)\right] \\
&\overset{(b)}{=} 4N\left[(-1)^{-\ell} X(-\ell \mod N) + (-1)^{-\ell} e^{-j\frac{2\pi\ell}{N}} X(\ell)\right] \\
&= 4N\left[(-1)^{\ell} X(-\ell \mod N) + (-1)^{\ell} e^{-j\frac{2\pi\ell}{N}} X(\ell)\right] \\
&= 4N(-1)^{\ell}\left[X(-\ell \mod N) + e^{-j\frac{2\pi\ell}{N}} X(\ell)\right]
\end{aligned}$$

where in step (a), we used the fact that $\frac{-4\ell \mod 4N}{4} = -\ell \mod N$. This result can be verified as follows; let $r = -4\ell \mod 4N$, then r can be written as $r = -4\ell - 4Nm$ where m is an integer, then $\frac{r}{4} = \frac{-4\ell-4Nm}{4} = -\ell - Nm = -\ell \mod N$.
In step (b), we used two facts; the first one is when N is even, $(-1)^{-\ell \mod N} = (-1)^{-\ell}$. This can be verified as follows. Let $r = -\ell \mod N = -\ell - Nm$ where m is an integer, then $(-1)^r = (-1)^{-\ell-Nm} = (-1)^{-\ell}$, since N is even. The second fact that we applied in (b) is that $e^{j\frac{2\pi(-\ell \mod N)}{N}} = e^{-j\frac{2\pi\ell}{N}}$. This was established in HW6 solutions.

4. (**10 PTS**) Given two causal sequences $a(n)$ and $b(n)$ defined over $0 \le n \le N-1$. Explain in words and clearly at least SEVEN different ways for computing the linear convolution $a(n) \star b(n)$. You receive zero credit on this problem if you do not list at least SIX valid ways.
Solution:
To evaluate $a(n) \star b(n)$, we have the following alternatives:

(1) Evaluate $a(n)\star b(n)$ analytically using the definition of linear convolutions: $\sum_{k=-\infty}^{\infty} a(k)b(n-k) = \sum_{k=-\infty}^{\infty} a(n-k)b(k)$;

(2) Evaluate $a(n) \star b(n)$ graphically where we fix one sequence and we flip and shift the second sequence. We then evaluate the product of the first sequence with the shifted version of the second sequence for each time shift;

(3) Evaluate $a(n) \star b(n)$ using the z-transform, i.e., $a(n) \star b(n) = \mathcal{Z}^{-1}(A(z).B(z))$ where $A(z)$ denotes the z-transform of $a(n)$ and $B(z)$ denotes the z-transform of $b(n)$;

(4) Evaluate $a(n)\star b(n)$ using DTFT, i.e., $a(n)\star b(n) = \text{IDTFT}(A(e^{j\omega}).B(e^{j\omega}))$ where $A(e^{j\omega})$ denotes the DTFT of $a(n)$, $B(e^{j\omega})$ denotes the DTFT of $b(n)$ and IDTFT stands for inverse DTFT.

(5) Evaluate $a(n)\star b(n)$ using circular convolution. This is done by extending $a(n)$ to $2N-1$ length sequence $a_{ext}(n)$ by appending to $a(n)$ $N-1$ zeros. We similarly extend $b(n)$ to $b_{ext}(n)$. Then $a(n) \star b(n) = a_{ext}(n) \circ b_{ext}(n)$. Therefore by evaluating analytically $a_{ext}(n) \circ b_{ext}(n)$, we obtain $a(n) \star b(n)$;

(6) Evaluate $a(n)\star b(n)$ by solving the circular convolution graphically. By evaluating graphically $a_{ext}(n) \circ b_{ext}(n)$, we obtain $a(n) \star b(n)$;

(7) Evaluate $a(n) \star b(n)$ by solving the circular convolution using DFT. By evaluating $a_{ext}(n) \circ b_{ext}(n) = \text{IDFT}(A_{ext}(k)B_{ext}(k))$, we obtain $a(n) \star b(n)$;

(8) Evaluate $a(n) \star b(n)$ by considering it as the output of the LTI system of impulse response $b(n)$ having as input $a(n)$. From $b(n)$, we can find the difference equation that describes the LTI system, then by solving this difference equation for the specific input $a(n)$, we obtain $a(n) \star b(n)$.