EE113: Digital Signal Processing Fall 2013
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SOLUTIONS FOR FINAL EXAMINATION

1. **(10 PTS; sampling theory)** Consider the following LTI system operating with a sampling frequency equal to F_s KHz:

$$y(n) = \frac{3}{4}y(n-1) + \frac{1}{2}x(n-1)$$

   (a) By how much will a tone at $F_s/4$ KHz be attenuated when filtered by this system? Is the attenuation dependent on the value of F_s?

   (b) If a tone at 4 KHz is attenuated by $2/\sqrt{13}$, can you tell what the sampling frequency F_s is?

Solution: Taking the DTFT of both sides of the difference equation, we obtain the frequency response of the LTI system:

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\frac{1}{2}e^{-j\omega}}{1-\frac{3}{4}e^{-j\omega}} = \frac{2e^{-j\omega}}{4-3e^{-j\omega}}$$

The magnitude response of the LTI system is then given by

$$\begin{aligned}
|H(e^{j\omega})| &= \left|\frac{2e^{-j\omega}}{4-3e^{-j\omega}}\right| \\
&= \frac{2}{|4-3[\cos(\omega)-j\sin(\omega)]|} \\
&= \frac{2}{|(4-3\cos(\omega))+j3\sin(\omega)|} \\
&= \frac{2}{\sqrt{(4-3\cos(\omega))^2+(3\sin(\omega))^2}} \\
&= \frac{2}{\sqrt{25-24\cos(\omega)}}
\end{aligned}$$

   (a) Now, for the tone at $F_s/4$ KHz, its digital angular frequency is given by

$$\omega = \frac{F_s/4}{F_s}\cdot 2\pi = \frac{\pi}{2}$$

   Therefore, the attenuation caused by the LTI system is given by

$$|H(e^{j\omega})|_{\omega=\pi/2} = \frac{2}{\sqrt{25-24\cos(\pi/2)}} = \frac{2}{5}$$

   It is obvious that the attenuation does *not* depend on the value of F_s.

(b) When the attenuation $|H(e^{j\omega})|$ is equal to 1/2, we have

$$\frac{2}{\sqrt{25-24\cos(\omega)}}=\frac{2}{\sqrt{13}}$$

or

$$\cos(\omega)=\frac{1}{2}$$

Therefore, the digital angular frequency of the tone that satisfies the above condition is given by

$$\omega=\arccos\left(\frac{1}{2}\right)=\frac{\pi}{3}$$

Since the corresponding analog frequency, denoted by F, is 4 KHz, the sampling frequency is given by

$$F_s=\frac{2\pi}{\omega}\cdot F=24 \ \text{KHz}$$

2. **(25 PTS; difference equations; z-transform)** A causal system is composed of the series cascade of two LTI systems with impulse response sequences given by the expressions

$$h_1(n)=\left(\frac{1}{2}\right)^{2n-1}u(2n-3),\qquad h_2(n)=\left(\frac{1}{2}\right)^{n}u(n-1)$$

(a) Determine the transfer function of the system.

(b) Determine the impulse response sequence of the system.

(c) Determine a model in terms of a constant-coefficient difference equation for the system. Denote its input and output sequences by $x(n)$ and $y(n)$, respectively.

(d) Is the system stable? What are its modes? zeros? poles?

(e) Assume the difference equation in part (c) is not relaxed. Determine initial conditions $y(-1)$ and $y(-2)$ such that only the largest mode appears at the output of the system when the input is $x(n)=\delta(n+3)$.

Solution:

(a) We can rewrite the impulse response sequences of $h_1(n)$ and $h_1(n)$ as

$$h_1(n)=\left(\frac{1}{2}\right)^{2n-1}u(2n-3)=\left(\frac{1}{2}\right)^{2n-1}u(n-2)=\frac{1}{8}\left(\frac{1}{4}\right)^{n-2}u(n-2)$$

and

$$h_2(n)=\left(\frac{1}{2}\right)^{n}u(n-1)=\frac{1}{2}\left(\frac{1}{2}\right)^{n-1}u(n-1)$$

Therefore, we have their z–transform as

$$H_1(z)=\frac{1}{8}\cdot\frac{z}{z-\frac{1}{4}}\cdot z^{-2},\qquad |z|>\frac{1}{4}$$

$$H_2(z) = \frac{1}{2} \cdot \frac{z}{z - \frac{1}{2}} \cdot z^{-1}, \quad |z| > \frac{1}{2}$$

The whole system is composed of the series cascade of $H_1(z)$ and $H_2(z)$. So the transfer function of the system is

$$H(z) = H_1(z) \cdot H_2(z) = \frac{1}{16} \cdot \frac{1}{(z - \frac{1}{4})(z - \frac{1}{2})z}, \quad |z| > \frac{1}{2}$$

(b) Using the partial fraction expansion, we have

$$H(z) = \frac{1}{16} \left(\frac{A}{z - \frac{1}{4}} + \frac{B}{z - \frac{1}{2}} + \frac{C}{z} \right)$$

where

$$A = \left. \frac{1}{(z - \frac{1}{2})z} \right|_{z=\frac{1}{4}} = -16, \quad B = \left. \frac{1}{(z - \frac{1}{4})z} \right|_{z=\frac{1}{2}} = 8, \quad C = \left. \frac{1}{(z - \frac{1}{4})(z - \frac{1}{2})} \right|_{z=0} = 8$$

Therefore, the impulse response sequence of the system is

$$h(n) = -\left(\frac{1}{4}\right)^{n-1} u(n-1) + \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} u(n-1) + \frac{1}{2}\delta(n-1)$$

(c) Let us denote the z–transform of $x(n)$ and $y(n)$ by $X(z)$ and $Y(z)$, respectively. From the transfer function

$$H(z) = \frac{1}{16} \cdot \frac{1}{(z - \frac{1}{4})(z - \frac{1}{2})z} = \frac{Y(z)}{X(z)}$$

Then,

$$16z^3 Y(z) - 12z^2 Y(z) + 2zY(z) = X(z)$$

Therefore we have the causal system with the constant-coefficient difference equation as

$$16y(n) - 12y(n-1) + 2y(n-2) = x(n-3)$$

(d) By solving the characteristic equation

$$16\lambda^2 - 12\lambda + 2 = 0$$

we get the modes

$$\lambda = \frac{1}{4}, \frac{1}{2}$$

The poles can be obtained by finding the roots of the denominator of $H(z)$:

$$\left(p - \frac{1}{4}\right)\left(p - \frac{1}{2}\right)p = 0$$

We then have the poles as

$$p = 0, \frac{1}{4}, \frac{1}{2}$$

The number of zeros is equal to the number of poles. Therefore, we have three zeros at the points $z = \pm\infty, \pm\infty, \pm\infty$.

The system is stable because it is causal and all modes have magnitude strictly less than one.

(e) Now we have the non-relaxed system that is described by the difference equation

$$16y(n) - 12y(n-1) + 2y(n-2) = x(n-3), \quad n \geq 0$$

with initial conditions $y(-1)$ and $y(-2)$. For $x(n) = \delta(n+3)$, we have

$$16y(n) - 12y(n-1) + 2y(n-2) = 0, \quad n \geq 1$$

and

$$16y(0) - 12y(-1) + 2y(-2) = 1$$

The general solution of the homogeneous equation has the form

$$y(n) = C_1 \left(\frac{1}{2}\right)^n + C_2 \left(\frac{1}{4}\right)^n$$

for some constants C_1 and C_2. Using the initial conditions $y(-1)$ and $y(-2)$, we get

$$\begin{cases} y(0) = \frac{1}{16} + \frac{3}{4}y(-1) - \frac{1}{8}y(-2) = C_1 + C_2 \\ y(-1) = 2C_1 + 4C_2 \end{cases}$$

If only the largest mode, which is $\lambda = \frac{1}{2}$, appears at the output of the system, then we must have $C_2 = 0$ and

$$4y(-1) - 2y(-2) + 1 = 0$$

and

$$y(-1) = 2C_1$$

for any constant $C_1 \neq 0$ for the non-relaxed system. Any initial conditions $y(-1)$ and $y(-2)$ satisfying the above equations are the solution. For example, we can pick $y(-1) = 1$ and $y(-2) = \frac{5}{2}$ and obtain

$$y(0) = \frac{1}{2}, \quad y(1) = \frac{1}{4}, \quad y(2) = \frac{1}{8}, \ldots$$

3. **(40 PTS; filters; frequency response; transfer function)** A sixth-order causal comb (LTI) filter is described by the constant-coefficient difference equation

$$y(n) = \alpha^6 y(n-6) + x(n) + x(n-6)$$

(a) Determine the filter transfer function.

(b) Determine the location of the zeros and poles of the filter.

(c) What are the modes of the system?

(d) Find conditions on α to ensure BIBO stability.

(e) Find the impulse response sequence of the filter.

(f) Determine the magnitude frequency response of the filter.

(g) Determine the step response of the filter.

(h) If the filter is operating at F_s KHz, by how much will a tone at $\frac{F_s}{12}$ KHz be attenuated?

Solution:

(a) The transfer function can be found by taking the z-transform of both sides of the difference equation:

$$Y(z) = \alpha^6 z^{-6} Y(z) + X(z) + z^{-6} X(z)$$

Solving for $H(z) = Y(z)/X(z)$, we obtain

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-6}}{1 - \alpha^6 z^{-6}}$$

(b) The zeros can be found by solving $z^6 + 1 = 0$. The result will be the roots of unity; the k-th zero is located at

$$\begin{aligned} z_k &= (-1)^{1/6} e^{j\frac{2\pi}{6}k} \\ &= e^{j\frac{\pi}{6}} e^{j\frac{2\pi}{6}k} \\ &= e^{j(\frac{2\pi}{6}k + \frac{\pi}{6})}, \qquad k = 0, \ldots, 5 \end{aligned}$$

The poles can be found by solving $z^6 - \alpha^6 = 0$. The k-th pole is located at

$$p_k = |\alpha| e^{j\frac{2\pi}{6}k}, \qquad k = 0, \ldots, 5$$

(c) The characteristic equation is given by

$$\lambda^6 - \alpha^6 = 0$$

Solving the characteristic equation, we have the the modes are located at

$$\lambda_\ell = |\alpha| e^{j\frac{2\pi}{6}\ell}$$

for $0 \leq \ell \leq 5$.

(d) Since the system is causal, in order for it to be stable, all modes must be strictly inside the unit-circle ($|\lambda_\ell| < 1$). This condition simplifies to

$$|\alpha| < 1$$

(e) To compute the impulse response of the filter, we compute the inverse z-transform of the transfer function $H(z)$. We already factored the transfer function as

$$\begin{aligned} H(z) &= \frac{\prod_{k=0}^{5}(z - z_k)}{\prod_{k=0}^{5}(z - p_k)} \\ &= \sum_{k=0}^{5} \frac{A_k}{z - p_k} \end{aligned}$$

where the coefficients A_k can be obtained by

$$A_k = H(z)(z - p_k)|_{z=p_k} = \frac{\prod_{\ell=0}^{5}(p_k - z_\ell)}{\prod_{\ell \neq k}^{5}(p_k - p_\ell)}$$

since the poles are distinct. Now, since the system is causal, we use the transform pair

$$p_k^{n-1}u(n-1) \longleftrightarrow \frac{1}{z-p_k}, \qquad |z| > |p_k|$$

to obtain

$$h(n) = \sum_{k=0}^{5} A_k p_k^{n-1} u(n-1)$$

(f) To obtain the frequency response of the filter, we simply evaluate $H(z)$ at $z = e^{j\omega}$:

$$H(e^{j\omega}) = \frac{1+e^{-6j\omega}}{1-\alpha^6 e^{-6j\omega}}$$

To find the magnitude response, we must compute $|H(e^{j\omega})|$:

$$\begin{aligned}
|H(e^{j\omega})| &= \sqrt{|H(e^{j\omega})|^2} \\
&= \sqrt{H(e^{j\omega})[H(e^{j\omega})^*]} \\
&= \sqrt{\frac{1+e^{-6j\omega}}{1-\alpha^6 e^{-6j\omega}} \cdot \frac{1+e^{6j\omega}}{1-\alpha^6 e^{6j\omega}}} \\
&= \sqrt{\frac{2+e^{-6j\omega}+e^{6j\omega}}{1-\alpha^6 e^{-6j\omega}-\alpha^6 e^{6j\omega}+\alpha^{12}}} \\
&= \sqrt{\frac{2+2\cos(6\omega)}{1-2\alpha^6\cos(6\omega)+\alpha^{12}}}
\end{aligned}$$

(g) To compute the step-response, we only need to convolve the impulse response with a step function:

$$\begin{aligned}
y_{\text{step}}(n) &= h(n) \star u(n) \\
&= \sum_{k=-\infty}^{\infty} \sum_{\ell=0}^{5} A_\ell p_\ell^{k-1} u(k-1)u(n-k) \\
&= \sum_{\ell=0}^{5} A_\ell \sum_{k=-\infty}^{\infty} p_\ell^{k-1} u(k-1)u(n-k) \\
&= \sum_{\ell=0}^{5} A_\ell p_\ell^{-1} \sum_{k=1}^{n} p_\ell^{k} \\
&= \sum_{\ell=0}^{5} A_\ell p_\ell^{-1} w(n)u(n)
\end{aligned}$$

where

$$w_\ell(n) \triangleq \begin{cases} \frac{1-p_\ell^{n+1}}{1-p_\ell} - 1, & p_\ell \neq 1 \\ n, & p_\ell = 1 \end{cases}$$

(h) Let $\Omega_0 = 2\pi\frac{F_s}{12}$. This frequency will map to $\omega_0 = \frac{\pi}{6}$. To compute by how much the tone will be attenuated, we need to compute the magnitude response of $H(e^{j\omega})$ and evaluate it at $\omega = \omega_0$. Before we evaluate the magnitude response, we will first compute $H(e^{j\omega_0})$. We have

$$H(e^{j\omega_0}) = \frac{0}{1-\alpha^6} = 0$$

since $e^{j\omega_0}$ corresponds to a zero of the transfer function. Therefore, the tone at $F_s/12$ will be completely eliminated.

4. **(10 PTS; DFT)** Consider a sequence $x(n)$ of length N, where N is even.

(a) What is the result of the following succession of operations on $x(n)$?

$$x(n) \overset{\text{DFT}}{\longrightarrow} \bullet \overset{(-j)^k}{\longrightarrow} \bullet \overset{\text{DFT}}{\longrightarrow} \bullet \overset{(-1)^n}{\longrightarrow} \bullet \overset{\text{IDFT}}{\longrightarrow} Y(k)$$

That is, $x(n)$ is first transformed by an N–point DFT, the result is modulated by the sequence $(-j)^k$, transformed by a second N–point DFT, modulated again by $(-1)^n$, and transformed one more time by the N–point inverse DFT.

(b) How does the energy of the output sequence $Y(k)$ relate to the energy of $x(n)$?

Solution: We first review some useful properties.

(i) Two consecutive DFTs: If

$$x(n) \overset{\text{DFT}}{\longrightarrow} X(k) \overset{\text{DFT}}{\longrightarrow} y(n)$$

then

$$y(n) = N \cdot x(-n \bmod N)$$

This property is proved as follows:

$$\begin{aligned}
y(n) &= \sum_{k=0}^{N-1} X(k) \cdot W_N^{kn} \\
&= \sum_{k=0}^{N-1} \left[\sum_{m=0}^{N-1} x(m) \cdot W_N^{mk} \right] \cdot W_N^{kn} \\
&= \sum_{m=0}^{N-1} x(m) \cdot \left[\sum_{k=0}^{N-1} W_N^{(m+n)k} \right] \\
&= \sum_{m=0}^{N-1} x(m) \cdot N \cdot \delta(m+n) \\
&= N \cdot x(-n \bmod N)
\end{aligned}$$

where $\delta(\cdot)$ denotes the Kronecker delta and

$$W_N \overset{\triangle}{=} e^{-j2\pi/N}$$

In the above derivation, we used the fact that

$$\sum_{k=0}^{N-1} W_N^{nk} = N \cdot \delta(n)$$

(ii) Two consecutive inverse DFTs: If

$$x(n) \stackrel{\text{IDFT}}{\longrightarrow} Z(k) \stackrel{\text{IDFT}}{\longrightarrow} z(n)$$

then

$$z(n) = \frac{1}{N} \cdot x(-n \bmod N)$$

This property is proved as follows:

$$\begin{aligned}
z(n) &= \frac{1}{N} \cdot \sum_{k=0}^{N-1} Z(k) \cdot W_N^{-kn} \\
&= \frac{1}{N} \cdot \sum_{k=0}^{N-1} \left[\frac{1}{N} \cdot \sum_{m=0}^{N-1} x(m) \cdot W_N^{-mk} \right] \cdot W_N^{-kn} \\
&= \frac{1}{N^2} \cdot \sum_{m=0}^{N-1} x(m) \cdot \left[\sum_{k=0}^{N-1} W_N^{-(m+n)k} \right] \\
&= \frac{1}{N^2} \cdot \sum_{m=0}^{N-1} x(m) \cdot N \cdot \delta(m+n) \\
&= \frac{1}{N} \cdot x(-n \bmod N)
\end{aligned}$$

(iii) DFT-modulation-DFT: If

$$x(n) \stackrel{\text{DFT}}{\longrightarrow} X(k) \stackrel{(-1)^k}{\longrightarrow} S(k) \stackrel{\text{DFT}}{\longrightarrow} w(n)$$

and N is even, then

$$w(n) = N \cdot x((-n + N/2) \bmod N)$$

This property is proved as follows:

$$\begin{aligned}
w(n) &= \sum_{k=0}^{N-1} S(k) \cdot W_N^{kn} \\
&= \sum_{k=0}^{N-1} X(k) \cdot (-1)^k \cdot W_N^{kn} \\
&= \sum_{k=0}^{N-1} X(k) \cdot W_N^{-kN/2} \cdot W_N^{kn} \\
&= \sum_{k=0}^{N-1} \left[\sum_{m=0}^{N-1} x(m) \cdot W_N^{mk} \right] \cdot W_N^{-kN/2} \cdot W_N^{kn} \\
&= \sum_{m=0}^{N-1} x(m) \cdot \left[\sum_{k=0}^{N-1} W_N^{(m+n-N/2)k} \right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=0}^{N-1} x(m) \cdot N \cdot \delta((m+n-N/2) \bmod N) \\
&= N \cdot x((-n+N/2) \bmod N)
\end{aligned}
$$

We also know that

$$x(n) \xrightarrow{\text{DFT}} X(k) \xrightarrow{\text{IDFT}} x(n)$$

(a) Let us denote the intermediate sequences as follows:

$$x(n) \xrightarrow{\text{DFT}} X(k) \xrightarrow{(-j)^k} S(k) \xrightarrow{\text{DFT}} z(n) \xrightarrow{(-1)^n} w(n) \xrightarrow{\text{IDFT}} Y(k)$$

where $X(k)$ is the DFT of $x(n)$. We investigate this chain reversely. First, from $w(n)$ to $Y(k)$, it is equivalent to

$$w(n) \xrightarrow{\text{DFT}} W(k) \xrightarrow{\text{IDFT}} w(n) \xrightarrow{\text{IDFT}} Y(k)$$

where $W(k)$ denotes the DFT of $w(n)$. Therefore, by using the property for two consecutive IDFTs, we have

$$Y(k) = \frac{1}{N} \cdot W(-k \bmod N)$$

Second, from $S(k)$ to $w(n)$ and then to $W(k)$, the relation is given by

$$S(k) \xrightarrow{\text{DFT}} z(n) \xrightarrow{(-1)^n} w(n) \xrightarrow{\text{DFT}} W(k)$$

Using the property for DFT-modulation-DFT, we have

$$W(k) = N \cdot S((-k+N/2) \bmod N)$$

Third, from $x(n)$ to $S(k)$, it is easy to verify that

$$S(k) = (-j)^k \cdot X(k)$$

Therefore, we get

$$Y(k) = S((k+N/2) \bmod N) = (-j)^{k+N/2} \cdot X((k+N/2) \bmod N)$$

(b) The energy of $Y(k)$ is given by

$$
\begin{aligned}
\sum_{k=0}^{N-1} |Y(k)|^2 &= \sum_{k=0}^{N-1} |(-j)^{k+N/2} \cdot X((k+N/2) \bmod N)|^2 \\
&= \sum_{k=0}^{N-1} |X((k+N/2) \bmod N)|^2 \\
&= \sum_{k=0}^{N-1} |X(k)|^2
\end{aligned}
$$

Using Parseval's relation, we have

$$\frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2 = \sum_{n=0}^{N-1} |x(n)|^2$$

Therefore,

$$\sum_{k=0}^{N-1} |Y(k)|^2 = N \cdot \sum_{n=0}^{N-1} |x(n)|^2$$

That is, the energy of $Y(k)$ is N-fold that of $x(n)$.

5. **(10 PTS; DFT)** Consider a sequence $x(n)$ of length N, where N is even.

   (a) If we apply the N–point DFT to $x(n)$ a total of $N/2$ times successively, what is the resulting sequence?

   (b) If we apply the N–point DFT to $x(-n \bmod N)$ a total of N times successively, what is the resulting sequence?

Solution: We start with the simple case. Applying one DFT to $x(n)$, we get $X(k)$. Applying two DFTs to $x(n)$, we get $N \cdot x(-n \bmod N)$. Applying three DFTs to $x(n)$ in a row is equivalent to applying two DFTs to $X(k)$, so we get $N \cdot X(-k \bmod N)$. Applying four DFTs to $x(n)$ in a row is equivalent to applying two DFTs to $N \cdot x(-n \bmod N)$, so we get $N^2 \cdot x(n)$, which is equal to the original sequence $x(n)$ apart from the scaling factor N^2. Therefore, we observe the following pattern:

$$\text{DFT}^m[x(n)] = \begin{cases} N^{2\ell} \cdot x(n), & m = 4\ell \\ N^{2\ell} \cdot X(k), & m = 4\ell + 1 \\ N^{2\ell+1} \cdot x(-n \bmod N), & m = 4\ell + 2 \\ N^{2\ell+1} \cdot X(-k \bmod N), & m = 4\ell + 3 \end{cases}$$

where we use $\text{DFT}^m[x(n)]$ to denote the result of m DFTs to $x(n)$ in a row and use ℓ to denote an integer.

(a) If we apply the N–point DFT to $x(n)$ a total of $N/2$ times successively, the resulting sequence will depend on the value of $N/2$:

$$\text{DFT}^{N/2}[x(n)] = \begin{cases} N^{2\ell} \cdot x(n), & N/2 = 4\ell \\ N^{2\ell} \cdot X(k), & N/2 = 4\ell + 1 \\ N^{2\ell+1} \cdot x(-n \bmod N), & N/2 = 4\ell + 2 \\ N^{2\ell+1} \cdot X(-k \bmod N), & N/2 = 4\ell + 3 \end{cases}$$

(b) Applying the N–point DFT to $x(-n \bmod N)$ a total of N times successively is equivalent to applying N–point DFT to $x(n)$ a total of $N + 2$ times successively and scaling by $1/N$. Therefore, the resulting sequence is given by

$$\begin{aligned} \text{DFT}^N[x(-n \bmod N)] &= \frac{1}{N} \cdot \text{DFT}^{N+2}[x(n)] \\ &= \begin{cases} N^{2\ell-1} \cdot x(n), & N+2 = 4\ell \\ N^{2\ell-1} \cdot X(k), & N+2 = 4\ell + 1 \\ N^{2\ell} \cdot x(-n \bmod N), & N+2 = 4\ell + 2 \\ N^{2\ell} \cdot X(-k \bmod N), & N+2 = 4\ell + 3 \end{cases} \end{aligned}$$

6. **(5 PTS; DTFT)** Evaluate the following series using properties of the DTFT:

$$\sum_{n=5}^{\infty} \frac{\cos^2(\pi n/4)}{n^2}$$

Solution: The series can be expressed by

$$\sum_{n=5}^{\infty} \frac{\cos^2(\pi n/4)}{n^2} = \sum_{n=1}^{\infty} \frac{\cos^2(\pi n/4)}{n^2} - \sum_{n=1}^{4} \frac{\cos^2(\pi n/4)}{n^2}$$

where

$$\sum_{n=1}^{\infty} \frac{\cos^2(\pi n/4)}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{\sin^2(\pi n/4)}{n^2}$$

and

$$\sum_{n=1}^{4} \frac{\cos^2(\pi n/4)}{n^2} = \frac{\cos^2(\pi/4)}{1} + \frac{\cos^2(\pi/2)}{4} + \frac{\cos^2(3\pi/4)}{9} + \frac{\cos^2(\pi)}{16} = \frac{89}{144}$$

Recall from expression (13.55) in Example 13.12 in the notes that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Moreover, introduce the sequence

$$x(n) = \frac{\sin^2(\pi n/4)}{n^2} = \frac{\pi^2}{16} \cdot \text{sinc}^2(\pi n/4)$$

Then, we have

$$\sum_{n=5}^{\infty} \frac{\cos^2(\pi n/4)}{n^2} = \frac{\pi^2}{6} - \frac{89}{144} - \sum_{n=1}^{\infty} x(n)$$

Since $x(n)$ is an even function of n, we have

$$\sum_{n=1}^{\infty} x(n) = \frac{1}{2}\left[\sum_{n=-\infty}^{\infty} x(n) - x(0)\right]$$

where

$$x(0) = \frac{\pi^2}{16} \cdot \text{sinc}^2(0) = \frac{\pi^2}{16}$$

Using Parseval's relation, the series $\sum_{n=-\infty}^{\infty} x(n)$ can be evaluated by

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x(n) &= \sum_{n=-\infty}^{\infty} \frac{\pi^2}{16} \cdot \text{sinc}^2(\pi n/4) \\
&= \pi^2 \cdot \sum_{n=-\infty}^{\infty} \left[\frac{1}{4} \cdot \text{sinc}(\pi n/4)\right] \cdot \left[\frac{1}{4} \cdot \text{sinc}(\pi n/4)\right]^* \\
&= \pi^2 \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{rect}\left(\frac{\omega}{\pi/4}\right) \cdot \left[\text{rect}\left(\frac{\omega}{\pi/4}\right)\right]^* d\omega
\end{aligned}$$

$$
\begin{aligned}
&= \pi^2 \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{rect}^2\left(\frac{\omega}{\pi/4}\right) d\omega \\
&= \pi^2 \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{rect}\left(\frac{\omega}{\pi/4}\right) d\omega \\
&= \pi^2 \cdot \frac{1}{2\pi} \cdot \frac{\pi}{2} \\
&= \frac{\pi^2}{4}
\end{aligned}
$$

where

$$
\operatorname{rect}\left(\frac{\omega}{\omega_c}\right) \triangleq \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| \geq \omega_c \end{cases}, \qquad \omega_c > 0
$$

Therefore, we end up with

$$
\sum_{n=5}^{\infty} \frac{\cos^2(\pi n/4)}{n^2} = \frac{\pi^2}{6} - \frac{89}{144} - \frac{1}{2}\left[\frac{\pi^2}{4} - \frac{\pi^2}{16}\right] = \frac{7\pi^2}{96} - \frac{89}{144} = \frac{21\pi^2 - 178}{288} \approx 0.101603
$$