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SOLUTIONS FOR FINAL EXAMINATION

(1) (5 PTS) If $H(z)$ is a rational transfer function that is causal and stable, what can you say about $H(z^{-1})$ and $H(-z)$?

Solution: Since $H(z)$ is causal and stable, all its poles are located inside the unit circle. Therefore, the ROC of $H(z)$ is given by

$$|z| > r$$

where

$$0 \leq r < 1$$

The ROC of $H(z^{-1})$ is then given by

$$|z^{-1}| > r$$

or

$$|z| < \frac{1}{r}$$

where

$$\frac{1}{r} > 1$$

Therefore, $H(z^{-1})$ represents an anti-causal and stable system. Similarly, the ROC of $H(-z)$ is given by

$$|-z| > r$$

or

$$|z| > r$$

which represents a causal and stable system.

(2) (5 PTS) Can an LTI system produce frequency components in the output sequence that are not present in the input sequence? Explain or give a counter-example when necessary.

Solution: No, it cannot. This is because in the frequency domain, the input-output relation is given by

$$Y(e^{j\omega}) = H(e^{j\omega}) \cdot X(e^{j\omega})$$

where $\{X(e^{j\omega}, H(e^{j\omega}, Y(e^{j\omega}\}$ are the DTFTs of the input sequence, the system impulse response sequence, and the output sequence, respectively. If

$$X(e^{j\omega}) = 0, \qquad \omega \in A$$

for some set A, then

$$Y(e^{j\omega}) = 0, \qquad \omega \in A$$

(3) (10 PTS) Let $x(n) = \alpha^n u(n)$ with $|\alpha| < 1$. Evaluate the following ratio by using the properties of the DTFT:

$$r = \frac{\sum_{n=0}^{\infty} n^2 x(n)}{\sum_{n=0}^{\infty} x(n)}$$

Solution: The DTFT of $x(n)$ is given by

$$X(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$$

The DTFT of $y(n) = n^2 x(n)$ is given by

$$Y(e^{j\omega}) = j\frac{d}{d\omega}\left[j\frac{d}{d\omega}X(e^{j\omega})\right] = \frac{\alpha(1 + \alpha e^{-j\omega})e^{-j\omega}}{(1 - \alpha e^{-j\omega})^3}$$

Then, we get

$$\sum_{n=0}^{\infty} x(n) = X(e^{j\omega})|_{\omega=0} = \frac{1}{1-\alpha}$$

and

$$\sum_{n=0}^{\infty} n^2 x(n) = Y(e^{j\omega})|_{\omega=0} = \frac{\alpha(1+\alpha)}{(1-\alpha)^3}$$

Therefore,

$$r = \frac{\frac{\alpha(1+\alpha)}{(1-\alpha)^3}}{\frac{1}{1-\alpha}} = \frac{\alpha(1+\alpha)}{(1-\alpha)^2}$$

(4) (20 PTS) The <u>even</u> part of the impulse response sequence of a causal LTI system is given by

$$h_e(n) = \left(\frac{1}{2}\right)^{|n|} - \left(\frac{1}{4}\right)^{|n|}$$

(a) Find the transfer function of the system.

(b) Find the unit-step response of the system.

(c) Find a constant-coefficient difference equation describing the system.

(d) Draw a block diagram representation for the system.

Solution: It can be verified that the impulse response sequence is given by

$$h(n) = 2h_e(n)u(n) - h_e(0)\delta(n) = 2\left[\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right]u(n)$$

(a) The transfer function is given by

$$H(z) = 2\left[\frac{z}{z-\frac{1}{2}} - \frac{z}{z-\frac{1}{4}}\right], \qquad |z| > \frac{1}{2}$$

(b) The z-transform of the unit-step sequence is given by

$$X(z) = \frac{z}{z-1}, \qquad |z| > 1$$

The z-transform of the unit-step response is then given by

$$Y(z) = H(z) \cdot X(z) = 2\left[\frac{z}{z-\frac{1}{2}} - \frac{z}{z-\frac{1}{4}}\right] \cdot \frac{z}{z-1} = \frac{\frac{4}{3}}{z-1} - \frac{1}{z-\frac{1}{2}} + \frac{\frac{1}{6}}{z-\frac{1}{4}}, \quad |z| > 1$$

Therefore, the unit-step response is given by

$$y(n) = \frac{4}{3}u(n-1) - \left(\frac{1}{2}\right)^{n-1} u(n-1) + \frac{1}{6}\left(\frac{1}{4}\right)^{n-1} u(n-1)$$

(c) The transfer function can be rewritten into

$$H(z) = \frac{\frac{1}{2}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}, \qquad |z| > \frac{1}{2}$$

It can be implemented by the following constant-coefficient difference equation:

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = \frac{1}{2}x(n-1)$$

(d) The system can be represented by the following diagram:

(5) (12 PTS) Consider the sequence $x(n) = (0.5)^n u(n)$. Evaluate the following quantities without explicitly finding the DTFT $X(e^{j\omega})$:

(a) $X(e^{j0})$.

(b) $X(e^{j\pi})$.

(c) $\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})d\omega$.

(d) $\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$.

Solution: The DTFT of $x(n)$ is defined by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

and the IDTFT of $X(e^{j\omega})$ is defined by

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

(a) By definition, we get

$$X(e^{j0}) = \sum_{n=-\infty}^{\infty} x(n) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u(n) = \frac{1}{1-\frac{1}{2}} = 2$$

(b) By definition, we get

$$X(e^{j\pi}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\pi n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u(n)(-1)^n = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$$

(c) By definition, we get

$$x(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j0n}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 1$$

(d) By the Parseval's relation, we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{2n} u(n) = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

(6) (8 PTS) Find a constant-coefficient difference equation to describe an LTI system whose impulse response sequence is given by

$$h(n) = \left(\frac{1}{2}\right)^{n-1} \sin\left(\frac{\pi n}{4}\right) u(n-2)$$

What is the frequency response of the system?

Solution: Let

$$h_o(n) = \left(\frac{1}{2}\right)^{n} \cos\left(\frac{\pi n}{4}\right) u(n)$$

The z-transform of $h_o(n)$ is given by

$$H_o(z) = \frac{z^2 - \frac{1}{2}z\cos\left(\frac{\pi}{4}\right)}{z^2 - 2\frac{1}{2}z\cos\left(\frac{\pi}{4}\right) + \left(\frac{1}{2}\right)^2} = \frac{z^2 - \frac{\sqrt{2}}{4}z}{z^2 - \frac{\sqrt{2}}{2}z + \frac{1}{4}}, \qquad |z| > \frac{1}{2}$$

Since

$$h(n) = \frac{1}{2}h_o(n-2)$$

Therefore, the z-transform of $h(n)$ is given by

$$H(z) = \frac{1}{2}z^{-2}H_o(z) = \frac{\frac{1}{2}z^2 - \frac{\sqrt{2}}{8}z}{z^4 - \frac{\sqrt{2}}{2}z^3 + \frac{1}{4}z^2} = \frac{\frac{1}{2}z^{-2} - \frac{\sqrt{2}}{8}z^{-3}}{1 - \frac{\sqrt{2}}{2}z^{-1} + \frac{1}{4}z^{-2}}, \qquad |z| > \frac{1}{2}$$

This system can be described by the following constant-coefficient difference equation:

$$y(n) - \frac{\sqrt{2}}{2}y(n-1) + \frac{1}{4}y(n-2) = \frac{1}{2}x(n-2) - \frac{\sqrt{2}}{8}x(n-3)$$

Since the ROC of the $H(z)$ contains the unit circle, the frequency response can be easily obtained and is given by

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = \frac{\frac{1}{2}e^{-j2\omega} - \frac{\sqrt{2}}{8}e^{-j3\omega}}{1 - \frac{\sqrt{2}}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega}}, \qquad \omega \in [-\pi, \pi]$$

(7) (15 PTS) What is the $N-$point DFT of the sequence

$$y(n) = \cos\left(\frac{2\pi k_o}{N} n\right) \cdot x(n) \ + \ (-1)^n \cdot x[(n-N) \bmod N]$$

in terms of the $N-$point DFT of $x(n)$? Assume that $x(n)$ has a finite length of N, $0 \leq n \leq N-1$, $0 \leq k_o \leq N-1$, and N is an even integer.

Solution: It is easy to verify

$$y(n) = \cos\left(\frac{2\pi k_o}{N} n\right) \cdot x(n) + e^{-j\frac{2\pi}{N} n \frac{N}{2}} \cdot x(n)$$

The N–point DFT of $y(n)$ is given by

$$Y(k) = \frac{1}{2}\left\{X[(k-k_o) \quad \mathrm{mod}\ N] + X[(k+k_o) \quad \mathrm{mod}\ N]\right\} + X\left[\left(k + \frac{N}{2}\right)\right]$$

(8) (10 PTS) Consider two periodic sequences $x(n)$ and $y(n)$ of periods N_x and N_y, respectively. Define $w(n) = x(n) + y(n)$.

(a) Show that $w(n)$ is periodic with period $N_x N_y$ at most.

(b) Determine the $N-$point DFT $W(k)$ in terms of the $N-$point DFTs $X(k)$ and $Y(k)$, where $N = N_x N_y$.

Solution:

(a) It is straightforward to verify

$$w(n + N_x N_y) = x(n + N_x N_y) + y(n + N_x N_y) = x(n) + y(n)$$

(b) By linearity, we get

$$W(k) = X(k) + Y(k)$$

(9) (15 PTS) A causal system is composed of the series cascade of two LTI subsystems with impulse response sequences $h_1(n) = \left(\frac{1}{2}\right)^n u(n)$ and $h_2(n) = \left(\frac{1}{3}\right)^n u(n-1)$. A tone at 1KHz is attenuated by $2/\sqrt{21}$ by the system (i.e., by approximately 43.6%). Can you tell what the sampling rate is?

Solution: The impulse response sequence of the overall system is given by

$$h(n) = h_1(n) \star h_2(n)$$

The corresponding frequency response is given by

$$H(e^{j\omega}) = H_1(e^{j\omega}) \cdot H_2(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} \cdot \frac{\frac{1}{3}e^{-j\omega}}{1 - \frac{1}{3}e^{-j\omega}} = \frac{\frac{1}{3}e^{-j\omega}}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{3}e^{-j\omega})}$$

When the attenuation for the tone at the normalized frequency ω_o is $2/\sqrt{21}$, we get the following equation

$$|H(e^{j\omega_o})| = \frac{2}{\sqrt{21}}$$

Using a calculator, it can be verified that

$$\omega_o = \frac{\pi}{3}$$

Therefore, the normalized frequency of this tone is $\pi/3$, which corresponding to 1 KHz. The sampling rate, denoted by Ω_s, is then given by

$$\Omega_s = \frac{2\pi}{\frac{\pi}{3}} = 6 \text{ KHz}$$