FINAL EXAM SOLUTIONS

1. (20 PTS) A causal system is described by the second-order difference equation

$$y(n) - \frac{7}{6}y(n-1) + \frac{1}{3}y(n-2) = x(n) - \frac{1}{2}x(n-1), \quad y(-1) = 1, \ y(-2) = 0$$

Find its complete response to $x(n) = \left(\frac{1}{2}\right)^n u(n)$:

(a) (10 PTS) Using the unilateral z–transform.

Take the unilateral z–transform of both sides of the difference equation

$$Y^+(z) - \frac{7}{6}\left[z^{-1}Y^+(z) + y(-1)\right] + \frac{1}{3}\left[z^{-2}Y^+(z) + z^{-1}y(-1) + y(-2)\right] =$$

$$X^+(z) - \frac{1}{2}\left[z^{-1}X^+(z) + x(-1)\right]$$

Substituting by the initial conditions, we get

$$Y^+(z)\left[1 - \frac{7}{6}z^{-1} + \frac{1}{3}z^{-2}\right] - \left[\frac{7}{6} - \frac{1}{3}z^{-1}\right] = X^+(z)\left[1 - \frac{1}{2}z^{-1}\right]$$

The unilateral z–transform of the input sequence $x(n) = \left(\frac{1}{2}\right)^n u(n)$ is given by

$$X^+(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

Then,

$$Y^+(z)\left[1 - \frac{7}{6}z^{-1} + \frac{1}{3}z^{-2}\right] = \frac{7}{6} - \frac{1}{3}z^{-1} + 1$$

and

$$Y^+(z) = \frac{\frac{13}{6} - \frac{1}{3}z^{-1}}{1 - \frac{7}{6}z^{-1} + \frac{1}{3}z^{-2}} = \frac{z\left(\frac{13}{6}z - \frac{1}{3}\right)}{z^2 - \frac{7}{6}z + \frac{1}{3}}$$

Using partial fractions, we can write $\frac{Y^+(z)}{z}$ as follows

$$\frac{Y^+(z)}{z} = \frac{\frac{13}{6}z - \frac{1}{3}}{z^2 - \frac{7}{6}z + \frac{1}{3}} = \frac{A}{z - \frac{2}{3}} + \frac{B}{z - \frac{1}{2}}$$

where A and B are given by

$$A = \left.\frac{Y^+(z)}{z}\left(z - \frac{2}{3}\right)\right|_{z=\frac{2}{3}} = \frac{20}{3}$$

$$B = \left.\frac{Y^+(z)}{z}\left(z - \frac{1}{2}\right)\right|_{z=\frac{1}{2}} = -\frac{9}{2}$$

Then

$$Y^+(z) = \frac{\frac{20}{3}z}{z - \frac{2}{3}} - \frac{\frac{9}{2}z}{z - \frac{1}{2}}$$

Using Inverse z–transform, the complete solution, $y(n)$, is given by

$$y(n) = \frac{20}{3}\left(\frac{2}{3}\right)^n u(n) - \frac{9}{2}\left(\frac{1}{2}\right)^n u(n)$$

(b) (**10 PTS**) Using the bilateral z–transform.

The complete solution can be obtained as the sum of the zero–input solution and the zero–state solution. Only the zero–state solution can be found using the z–transform whereas the zero–input solution has to be found by substituting the initial conditions into the homogeneous solutions.

– Zero–state solution
   We consider the relaxed system

$$y(n) - \frac{7}{6}y(n-1) + \frac{1}{3}y(n-2) = x(n) - \frac{1}{2}x(n-1), \quad y(-1) = y(-2) = 0$$

We find its response to $x(n) = \left(\frac{1}{2}\right)^n u(n)$. Using z–transform, we get

$$Y(z) - \frac{7}{6}z^{-1}Y(z) + \frac{1}{3}z^{-2}Y(z) = X(z) - \frac{1}{2}z^{-1}X(z)$$

The z–transform of the input sequence is given by

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

Then,

$$Y(z) = \frac{1}{1 - \frac{7}{6}z^{-1} + \frac{1}{3}z^{-2}} = \frac{z^2}{z^2 - \frac{7}{6}z + \frac{1}{3}}$$

Using partial fractions,

$$\frac{Y(z)}{z} = \frac{z}{z^2 - \frac{7}{6}z + \frac{1}{3}} = \frac{C}{z - \frac{2}{3}} + \frac{D}{z - \frac{1}{2}}$$

where C and D are given by

$$C = \left.\frac{Y(z)}{z}\left(z - \frac{2}{3}\right)\right|_{z=\frac{2}{3}} = 4$$

$$D = \left.\frac{Y(z)}{z}\left(z - \frac{1}{2}\right)\right|_{z=\frac{1}{2}} = -3$$

Then

$$Y(z) = \frac{4z}{z - \frac{2}{3}} - \frac{3z}{z - \frac{1}{2}}$$

Using Inverse z–transform, the zero–state solution, $y_{zs}(n)$, is given by

$$y_{zs}(n) = 4 \left(\frac{2}{3}\right)^n u(n) - 3 \left(\frac{1}{2}\right)^n u(n)$$

– Zero–input solution

The characteristic equation of the system is

$$\lambda^2 - \frac{7}{6}\lambda + \frac{1}{3} = 0$$

$$\left(\lambda - \frac{2}{3}\right)\left(\lambda - \frac{1}{2}\right) = 0$$

Then the zero–input solution is

$$y_{zi}(n) = C_1 \left(\frac{2}{3}\right)^n + C_2 \left(\frac{1}{2}\right)^n$$

We then use the initial conditions $y(-1) = 1$ and $y(-2) = 0$ to find C_1 and C_2 as follows

$$y(-1) = 1 = \frac{3}{2}C_1 + 2C_2$$

$$y(-2) = 0 = \frac{9}{4}C_1 + 4C_2$$

We get $C_1 = \frac{8}{3}$ and $C_2 = -\frac{3}{2}$. Then,

$$y_{zi}(n) = \frac{8}{3} \left(\frac{2}{3}\right)^n u(n) - \frac{3}{2} \left(\frac{1}{2}\right)^n u(n)$$

– The complete solution is now given by

$$y(n) = y_{zs}(n) + y_{zi}(n) = \frac{20}{3} \left(\frac{2}{3}\right)^n u(n) - \frac{9}{2} \left(\frac{1}{2}\right)^n u(n)$$

2. **(20 PTS)** A relaxed, causal, and stable system is described by the first-order difference equation

$$y(n) - ay(n-1) = x(n)$$

where $x(n)$ denotes the input sequence and $y(n)$ denotes the output sequence.

(a) **(10 PTS)** Find the value of the scalar a in order to guarantee that a unit-amplitude tone at 750Hz that is sampled at twice its Nyquist rate is attenuated by $2/\sqrt{7}$.

The Nyquist rate is twice the maximum frequency component in the sampled signal then the sampling frequency is given by

$$F_s = 2 \times 2 \times 750 = 3\text{kHz}$$

If the input tone is given by $x(t) = \cos(2\pi \times 750t)$, then the sampled signal is given by

$$x(n) = x(t)|_{t=n/F_s} = \cos(2\pi \times \frac{750}{3000}n) = \cos(\frac{\pi}{2}n)$$

when $x(n)$ is applied to a system with real impulse response sequence, the output $y(n)$ is given by

$$y(n) = \left|H(e^{j\frac{\pi}{2}})\right| \cos\left(\frac{\pi}{2}n + \angle H(e^{j\frac{\pi}{2}})\right)$$

The attenuation of the system is then given by its magnitude response at $\omega = \frac{\pi}{2}$. To find the magnitude response of the system we proceed as follows:

– Take the DTFT of both sides of the difference equation

$$Y(e^{j\omega}) - ae^{-j\omega}Y(e^{j\omega}) = X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - ae^{-j\omega}} = \frac{1}{1 - a\cos\omega + j\sin\omega}$$

– The magnitude response is then given by

$$\left|H(e^{j\omega})\right| = \frac{1}{\sqrt{(1 - a\cos\omega)^2 + \sin^2\omega}} = \frac{1}{\sqrt{1 + a^2 - 2a\cos\omega}}$$

– At $\omega = \frac{\pi}{2}$, we have

$$\left|H(e^{j\frac{\pi}{2}})\right| = \frac{1}{\sqrt{1 + a^2}} = \frac{2}{\sqrt{7}}$$

Then,

$$a^2 + 1 = \frac{7}{4} \quad \Longrightarrow \quad a^2 = \frac{3}{4} \quad \Longrightarrow \quad \boxed{a = \pm\frac{\sqrt{3}}{2}}$$

(b) **(5 PTS)** Find all values of the scalar a so that the energy of the impulse response sequence of the system is equal to $4/3$.

We first use the z–transform to find the transfer function of the system, $H(z)$, as follows

$$Y(z) - az^{-1}Y(z) = X(z) \quad \Longrightarrow \quad H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - az^{-1}}$$

We then use the Inverse z–transform to get the impulse response sequence

$$h(n) = a^n u(n)$$

The energy of $h(n)$ is given by

$$\mathcal{E}_h = \sum_{n=-\infty}^{\infty} |h(n)|^2 = \sum_{n=0}^{\infty} a^{2n} = \sum_{n=0}^{\infty} (a^2)^n = \frac{1}{1-a^2}$$

where $a < 1$ since the system is stable.

$$\frac{1}{1-a^2} = \frac{4}{3} \quad \implies \quad a^2 = \frac{1}{4} \quad \implies \quad \boxed{a = \pm\frac{1}{2}}$$

(c) (**5 PTS**) Find the value of the scalar a so that the response of the system to $x(n) = u(n)$ is

$$y(n) = \left[2 - \left(\frac{1}{2}\right)^n\right] u(n)$$

We use the z–transform to find the transfer function of the system using the given input–output pair and we compare it to the expression we obtained in part (b) to find the value of the scalar a,

$$x(n) = u(n) \qquad \implies \qquad X(z) = \frac{z}{z-1}, \ |z| > 1$$

$$y(n) = \left[2 - \left(\frac{1}{2}\right)^n\right] u(n) \qquad \implies \qquad Y(z) = \frac{2z}{z-1} - \frac{z}{z-\frac{1}{2}}, \ |z| > 1$$

$$Y(z) = \frac{z^2}{(z-1)(z-\frac{1}{2})}, \ |z| > 1$$
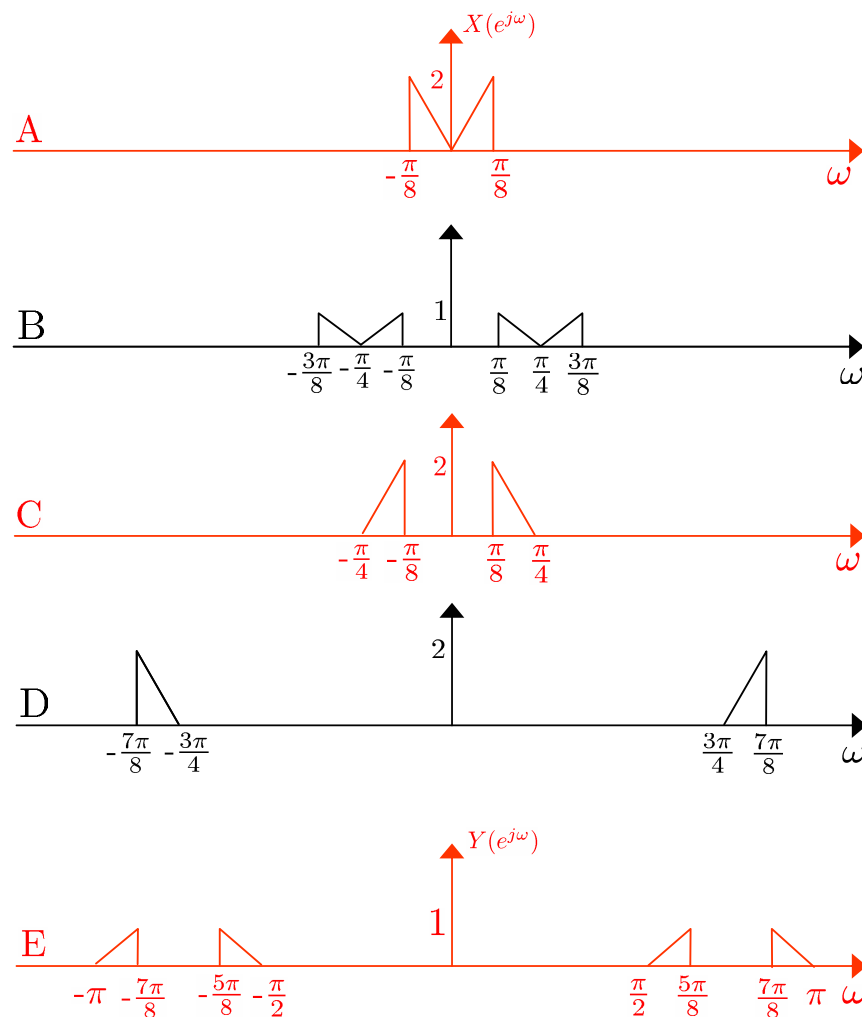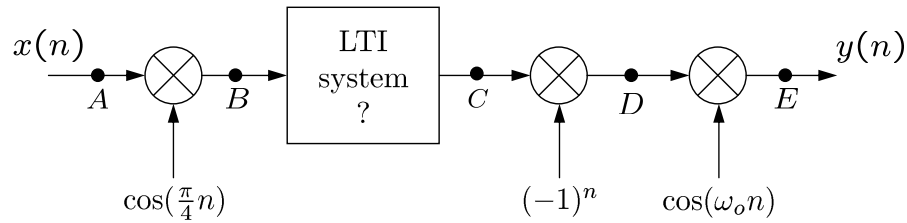
Then

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z}{z-\frac{1}{2}} = \frac{1}{1-\frac{1}{2}z^{-1}}, \ |z| > \frac{1}{2}$$

Comparing this expression to $H(z)$ from part (b), we get

$$\boxed{a = \frac{1}{2}}$$

3. (30 PTS) Consider the block diagram shown in the figure below where the LTI system is a lowpass filter. The DTFTs of the sequences at the points A, C, and E are also shown.

(a) (10 PTS) Find the frequency response and the impulse response of the unknown LTI system. Is the LTI system causal?

By comparing the plots at the input and output of the unknown LTi system (points A and B, respectively), we find that the system filters out all frequency components with

frequencies greater than $\pi/4$. Moreover, the output magnitude is twice that of the input. Hence, the frequency response of the LTI system is then given by

$$H(e^{j\omega}) = \begin{cases} 2 & |\omega| \leq \frac{\pi}{4} \\ 0 & \frac{\pi}{4} < |\omega| \leq \pi \end{cases}$$

and the impulse response sequence is

$$h(n) = \begin{cases} \dfrac{1}{2} & n = 0 \\ 2\dfrac{\sin(\frac{\pi}{4})n}{\pi n} & n \neq 0 \end{cases}$$

Since $h(n) \neq 0$ for $n < 0$, then the system is <u>not causal</u>.

(b) **(5 PTS)** Plot the DTFT of the sequences at points B and D.

The plots are shown in the figure above.

(c) **(5 PTS)** Find ω_o.

Since the multiplication by $\cos(\omega_o n)$ results in shifting the DTFT to the left and to the right by ω_o and scaling by $\frac{1}{2}$, we conclude from the plot at points D and E that $\omega_o = \frac{\pi}{4}$

(d) **(5 PTS)** Evaluate $x(n) \star y(n)$.

We know that

$$x(n) \star y(n) \xleftarrow{DTFT} X(e^{j\omega})Y(e^{j\omega})$$

From the plot at points A and E we get

$$X(e^{j\omega})Y(e^{j\omega}) = 0$$

Therefore

$$x(n) \star y(n) = 0$$

(e) **(5 PTS)** Find the energy of the output sequence $y(n)$.

Using Parsaval's theorem, the energy of $y(n)$ is given by

$$\mathcal{E}_y = \sum_{n=-\infty}^{\infty} |y(n)|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

which is equal to four times the area under the square of a triangle of a base width $\frac{\pi}{8}$ and a hight of 1. i.e.,

$$\mathcal{E}_y = 4 \times \frac{1}{2\pi} \int_0^{\frac{\pi}{8}} \left(\frac{8\omega}{\pi}\right)^2 d\omega = \frac{1}{12}$$

4. (30 PTS) Let $x(n)$ be the N–point sequence $\{x(0), x(1), \cdots, x(N-1)\}$ and let $X(k)$ denote its N–point DFT sequence. Define the sequence $x_1(n)$ of length $2N$ that is constructed from $x(n)$ as follows:

$$x_1(n) = \{x(0),\ 0,\ x(1),\ 0,\ \cdots\cdots,\ 0,\ x(N-1),\ 0\}$$

In other words, a zero is added following each sample of $x(n)$. This operation is known as interpolation and it is usually indicated in block diagram form as follows:

Define also the extended DFT sequence

$$X_2(k) = \{\underbrace{X(0),\ \cdots,\ X(N-1)}_{X(k)},\ \underbrace{0,\ \cdots\cdots,\ 0}_{N\ \text{zeros}}\}$$

That is, N zeros are appended to $X(k)$.

(a) (10 PTS) Find the $2N$–point DFT of $x_1(n)$ in terms of the samples of $X(k)$.

The sequence $x_1(n)$ can be written as

$$x_1(n) = \begin{cases} x(n/2) & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases}$$

The $2N$–point DFT of $x_1(n)$ is given by

$$X_1(k) = \sum_{n=0}^{2N-1} x_1(n) e^{-j\frac{2\pi}{2N}nk}, \qquad k = 0, \cdots, 2N-1 \tag{1}$$

$$= \sum_{\substack{n=0, \\ n \text{ even}}}^{2N-1} x(\tfrac{n}{2}) e^{-j\frac{2\pi}{2N}nk}, \qquad k = 0, \cdots, 2N-1 \tag{2}$$

Let $m = n/2$, then

$$X_1(k) = \sum_{m=0}^{N-\frac{1}{2}} x(m) e^{-j\frac{2\pi}{N}mk}, \qquad k = 0, \cdots, 2N-1$$

Since m takes only integer values, we can change the upper limit of the sum to $N-1$. Therefore,

$$X_1(k) = \sum_{m=0}^{N-1} x(m) e^{-j\frac{2\pi}{N}mk} = X(k), \qquad k = 0, \cdots, 2N-1$$

Note that k takes values between 0 and $2N - 1$. Since X(k) is a periodic sequence with period N, it repeats itself between N and $2N - 1$. This leads to

$$X_1(k) = \{X(k), X(k)\}$$

where $X(k)$ is the N point DFT of $x(n)$.

(b) (**10 PTS**) Find the even samples of the inverse 2N- point DFT of $X_2(k)$ in terms of $x(n)$.

The inverse $2N$–point DFT of $X_2(k)$ is given by

$$
\begin{aligned}
x_2(n) &= \frac{1}{2N} \sum_{k=0}^{2N-1} X_2(k) e^{j\frac{2\pi}{2N}nk}, \qquad n = 0, \cdots, 2N - 1 \\
&= \frac{1}{2N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{2N}nk}, \qquad n = 0, \cdots, 2N - 1
\end{aligned}
$$

For even values of N,

$$
\begin{aligned}
x_2(n) &= \frac{1}{2} \left(\frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}\left(\frac{n}{2}\right)k} \right), \qquad n = 0, 2, \cdots, 2N - 2 \\
&= \frac{1}{2} x\left(\frac{n}{2}\right), \qquad n = 0, 2, \cdots, 2N - 2
\end{aligned}
$$

(c) (**10 PTS**) Let $x(n)$ and $y(n)$ be N–point sequences with N–point DFTs $X(k)$ and $Y(k)$, respectively. Let $z(n)$ be the output of the block diagram shown below. Express $z(n)$ in terms of $x(n)$ and $y(n)$.
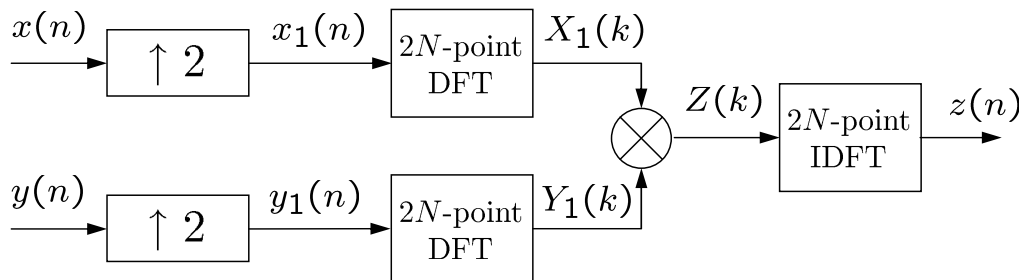
The $2N$–point DFT of $x_1(n)$ is shown in part (a) to be

$$X_1(k) = \{X(k), \ X(k)\}$$

where $X(k)$ is the N–point DFT of $x(n)$. Similarly, The $2N$–point DFT of $y_1(n)$ is

$$Y_1(k) = \{Y(k), \ Y(k)\}$$

Then,
$$Z(k) = X_1(k)Y_1(k) = \{X(k)Y(k),\ X(k)Y(k)\}$$

Let $A(k) = X(k)Y(k)$, and let $a(n)$ be the N-point IDFT of $A(k)$. Since $Z(k) = \{A(k),\ A(k)\}$ then the $2N$–point IDFT of $Z(k)$ is the interpolation of $a(n)$. i.e.,

$$
\begin{aligned}
z(n) &= \begin{cases} a(n/2) & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases} \\
&= \{a(0),\ 0,\ a(1),\ 0,\ \cdots\cdots,\ 0,\ a(N-1),\ 0\}
\end{aligned}
$$

It remains now to find the samples of the sequence $a(n)$, that is the N–point IDFT of the product of $X(k)$ and $Y(k)$. This is clearly the circular convolution of $x(n)$ and $y(n)$.

$$a(n) = \text{IDFT}\{X(k)Y(k)\} = x(n) \circ y(n)$$