**EE113: Digital Signal Processing**
Prof. A. H. Sayed | Dec. 12, 2002

# FINAL EXAMINATION

(Open Book)

1. (10 PTS) For the block diagram shown in Figure 1:

   (a) Find the system transfer function, $H(z)$.

   (b) Let $x(t) = \cos(100\pi t) + \cos(500\pi t) + \cos(900\pi t)$ and assume it is sampled at $f_s = 1000$Hz to generate $x(n)$. Find the attenuation of each tone after passing through $H(z)$.

   (c) Is $H(z)$ a lowpass or a highpass filter?

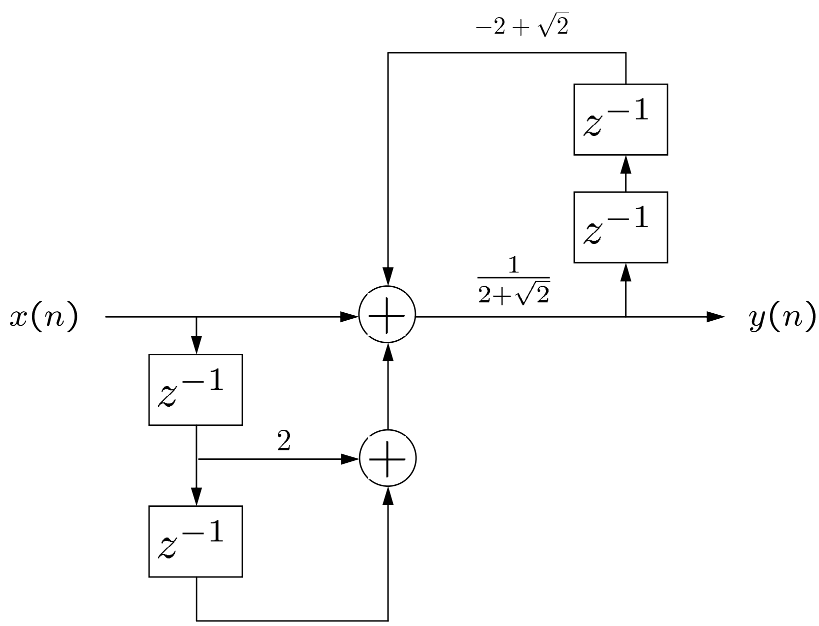


Figure 1: A discrete–time filter.

2. (10 PTS) Let $x(n)$ be an $N$-sample sequence with $N$-point DFT $X(k)$, with $N$ even. Let $x_e(n)$ and $x_o(n)$ be the even and odd samples of $x(n)$, respectively, i.e.,

$$
\begin{aligned}
x_e(n) &= \{x(0), x(2), \cdots, x(N-2)\} \\
x_o(n) &= \{x(1), x(3), \cdots, x(N-1)\}
\end{aligned}
$$

   Let also $X_e(k)$ and $X_o(k)$ denote the $\frac{N}{2}$-point DFTs of $x_e(n)$ and $x_o(n)$, respectively. Express the $N$ samples of $X(k)$ in terms of the $\frac{N}{2}$ samples of $X_e(k)$ and $X_o(k)$.

3. (20 PTS) Refer to Figure 2. It shows a system $H_1(z) = \frac{1}{z+\frac{3}{2}}$ in cascade with an unknown LTI system. Both systems are assumed initially relaxed.
   (a) Is the overall system LTI ? Why?
   (b) Find the transfer function of the unknown LTI system.

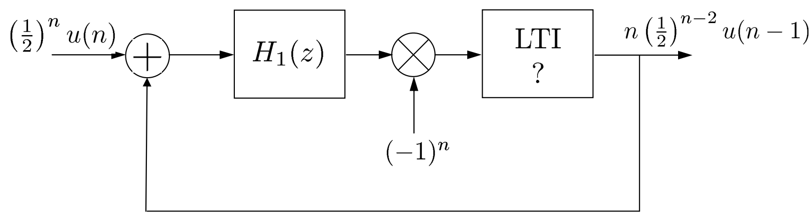


Figure 2: A Feedback system.

4. (30 PTS) Figure 3 shows the DTFT of two sequences $x(n)$ and $y(n)$.
   (a) Express $y(n)$ in terms of $x(n)$.
   (b) Refer to Figure 4. Find the cut–off frequencies $\omega_1$ and $\omega_2$ so that the block diagram shown in the figure can be used to recover $x(n)$ from $y(n)$.
   (c) Find the frequency response $H(e^{j\omega})$ of the block indicated by dotted lines. Is it a lowpass or a highpass filter?
   (d) Plot the DTFTs of the signals at the points (A,B,C,D,E) of Figure 4.

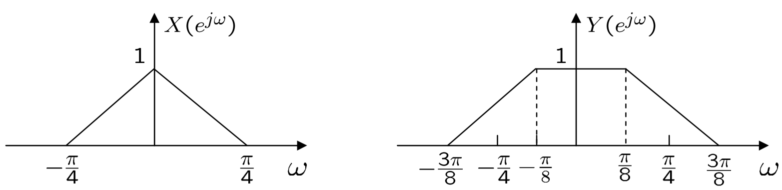


Figure 3: The DTFT of two sequences $x(n)$ and $y(n)$.

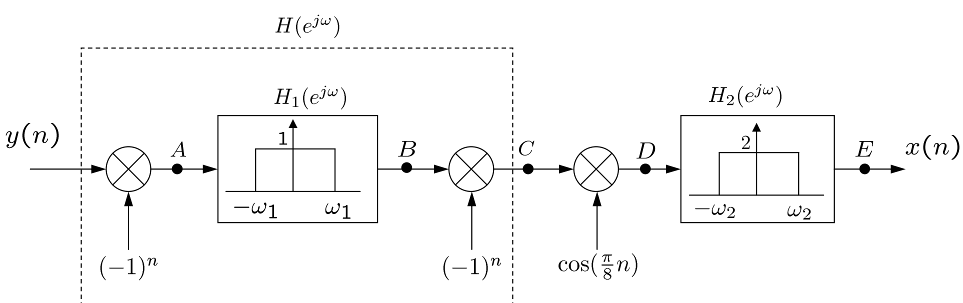


Figure 4: Processing of $y(n)$ by discrete–time filters.

5. (30 PTS) For the block diagram shown in Figure 5:
   (a) Find $y(n)$.
   (b) Plot $X_1(e^{j\omega})$ and $Y(e^{j\omega})$.
   (c) Evaluate the energy of $y(n)$.
   (d) Compute the 5-point DFT of $y(n)$.

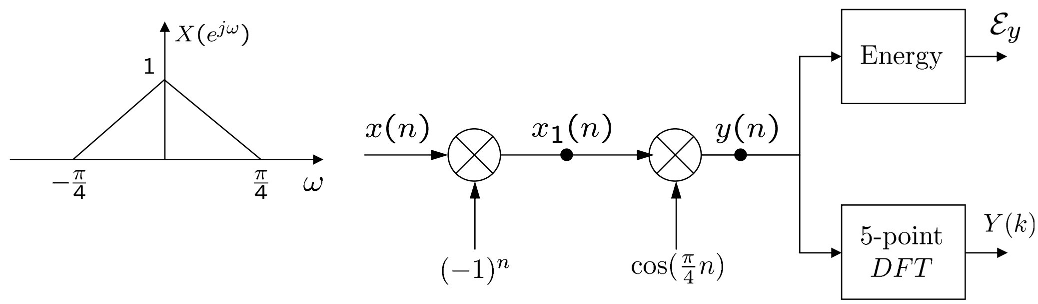


Figure 5: Block diagram for problem 5.

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1. (a) From Fig. 1

$$[X(z)+2z^{-1}X(z)+z^{-2}X(z)+z^{-2}Y(z)(-2+\sqrt{2})]/(2+\sqrt{2})=Y(z)$$

$$X(z)(1+2z^{-1}+z^{-2})=Y(z)[(2+\sqrt{2})+(2-\sqrt{2})z^{-2}]$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{1+2z^{-1}+z^{-2}}{(2+\sqrt{2})+(2-\sqrt{2})z^{-2}}$$

(b) $f_s=1000\,Hz$ , $T_s=1/f_s=10^{-3}\,s$

$$x(n)=\cos(0.1\pi n)+\cos(0.5\pi n)+\cos(0.9\pi n)$$

Notice, there is no aliasing in frequency

$f_s=1000\,Hz > 2\times 50\,Hz,\ 2\times 250\,Hz,\ 2\times 450\,Hz$.

$\cos(0.1\pi n)$

$\Rightarrow |H(e^{j0.1\pi})|\cos[0.1\pi n+\measuredangle H(e^{j0.1\pi})]$

attenuation $=|H(e^{j0.1\pi})|$

$$\left|\{H(z)|_{z=e^{j\omega}}\}\right|=\frac{|1+2e^{-j\omega}+e^{-2j\omega}|}{|(2+\sqrt{2})+(2-\sqrt{2})e^{-2j\omega}|}$$

$$=\frac{|1+e^{-j\omega}|^2}{|(2+\sqrt{2})+(2-\sqrt{2})\cos 2\omega-j(2-\sqrt{2})\sin 2\omega|}$$

$$=\frac{(2+2\cos\omega)}{\sqrt{12+4\cos 2\omega}}$$

$$=\frac{1+\cos\omega}{\sqrt{3+\cos 2\omega}}$$

ⅰ. $\cos(0.1\pi n)$

$\Rightarrow$ attenuation $= |H(e^{j0.1\pi})| = 0.9996855$

$\cos(0.5\pi n)$

$\Rightarrow$ att $= |H(e^{j0.5\pi})| = 0.7071068$

$\cos(0.9\pi n)$

$\Rightarrow$ att $= |H(e^{j0.9\pi})| = 0.0250777$

(c) From (b), we can see it is a ~~se~~ low pass filter.

Actually, it is a second order Butterworth low pass filter.

2. Define phase factor

$$W_N \triangleq e^{-j\frac{2\pi}{N}} = \cos\left(\frac{2\pi}{N}\right) - j\sin\left(\frac{2\pi}{N}\right)$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$= \sum_{n=\text{even}} x(n) W_N^{kn} + \sum_{n=\text{odd}} x(n) W_N^{kn}$$

$$= \sum_{m=0}^{\frac{N}{2}-1} x(2m) W_N^{k2m} + \sum_{m=0}^{\frac{N}{2}-1} x(2m+1) W_N^{k(2m+1)}$$

$$= \sum_{m=0}^{\frac{N}{2}-1} x(2m) W_{N/2}^{km} + \sum_{m=0}^{\frac{N}{2}-1} x(2m+1) W_{N/2}^{km} \cdot W_N^{k}$$

$$= \underbrace{\sum_{m=0}^{\frac{N}{2}-1} x_e(m) W_{N/2}^{km}}_{X_e(k)} + W_N^{k} \underbrace{\sum_{m=0}^{\frac{N}{2}-1} x_o(m) W_{N/2}^{km}}_{X_o(k)}$$

$X_e(k)$ — $\frac{N}{2}$ point DFT of $x_e(n)$

$X_o(k)$ — $\frac{N}{2}$ point DFT of $x_o(n)$

$X(k) = X_e(k) + W_N^k X_o(k)$

Notice $W_N^{k+\frac{N}{2}} = -W_N^k$, we have

$$\begin{cases} X(k) = X_e(k) + W_N^k X_o(k) & 0 \le k \le \frac{N}{2}-1 \\ X(k+\frac{N}{2}) = X_e(k) - W_N^k X_o(k) \end{cases}$$

If N is a power of 2, i.e. $N = 2^k$, above computation can be recursively carried out until the evaluation of 2-point DFT. And it is call FFT (Fast Fourier Transform).

3\. (a) No, it is not LTI.

Because $y(n) = (-1)^n x(n)$ is not time-invariant

(b) $x(n) = (\frac{1}{2})^n u(n)$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{z}{(z-\frac{1}{2})}$$

$$y(n) = n(\tfrac{1}{2})^{n-2} u(n-1) = 2n(\tfrac{1}{2})^{n-1} u(n-1)$$

$$Y(z) = -2z \frac{d}{dz}\left[\frac{1}{(z-\frac{1}{2})}\right] = \frac{2z}{(z-\frac{1}{2})^2}$$

Or: $y(n) = 2(n-1)(\frac{1}{2})^{n-1} u(n-1) + 2(\frac{1}{2})^{n-1} u(n-1)$

$$Y(z) = \frac{1}{(z-\frac{1}{2})^2} + \frac{2}{(z-\frac{1}{2})} = 2z/(z-\tfrac{1}{2})^2$$

$$Y_1(z) = X(z) + Y(z) = \frac{z}{(z-\frac{1}{2})} + \frac{2z}{(z-\frac{1}{2})^2}$$
$$= \frac{z^2 + 3z/2}{(z-\frac{1}{2})^2} = \frac{z(z+3/2)}{(z-\frac{1}{2})^2}$$

$$Y_2(z) = Y_1(z) \cdot H_1(z)$$
$$= \frac{z}{(z-\frac{1}{2})^2}$$

$$y_2(n) = 2n(\tfrac{1}{2})^n u(n)$$

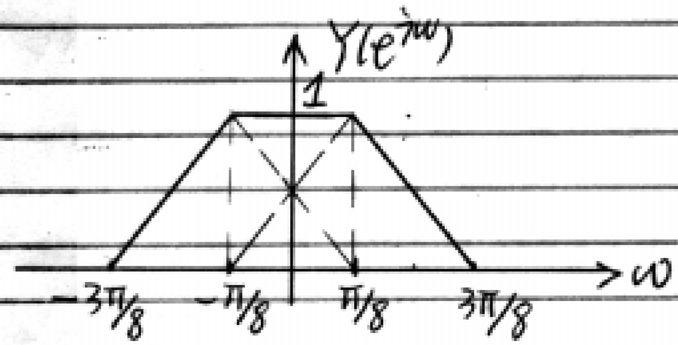
$$(-1)^n y_2(n) = 2n(-\tfrac{1}{2})^n u(n)$$
$$= y_3(n)$$

$$Y_3(z) = -\frac{z}{(z+\frac{1}{2})^2}$$

unknow LTI system

$$H_2(z) = \frac{Y(z)}{Y_3(z)} = -\frac{2(z+\frac{1}{2})^2}{(z-\frac{1}{2})^2}$$

4. (a)




Shift $X(e^{j\omega})$ to left and right by $\pi/8$, and add.
This results on $Y(e^{j\omega})$ $\Longrightarrow$ modulation

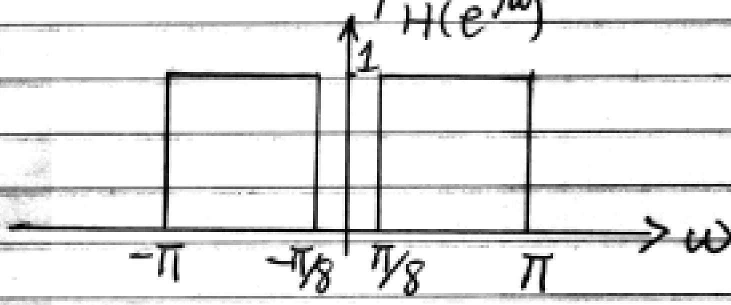
$\therefore$ $y(n) = x(n)\cos(\frac{\pi}{8}n)$

(b) $\omega_1 = 7\pi/8$

$\omega_2 = \pi/4$

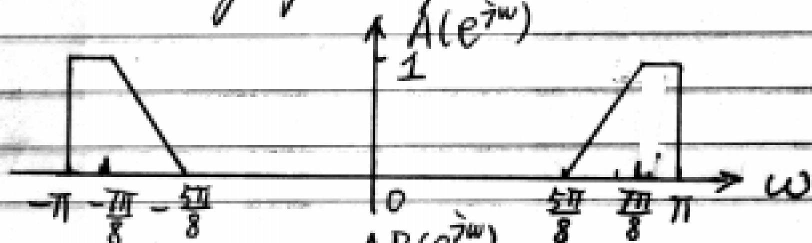
This becomes clear in (d)

(c) Use $\delta(n)$ as input

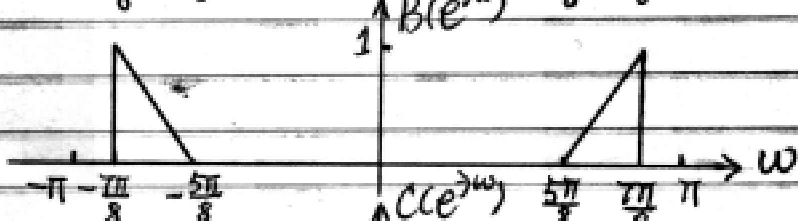



It is a highpass filter.

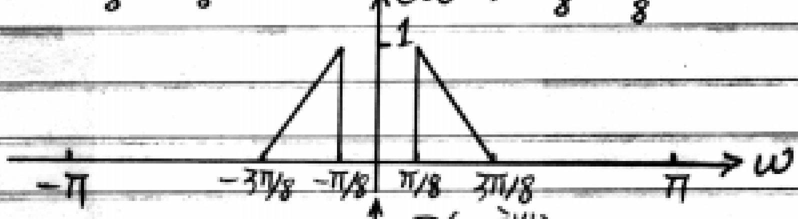
(d)



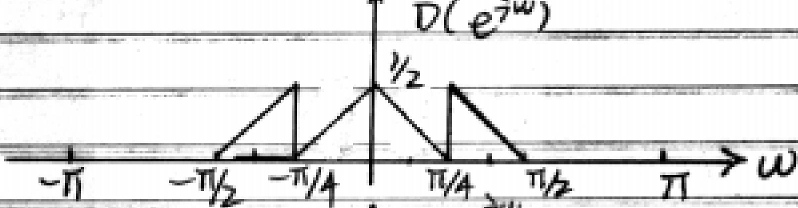

$\leftarrow A(e^{j\omega})$



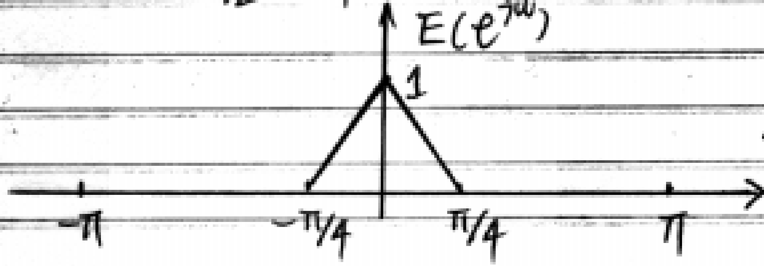

$\leftarrow B(e^{j\omega})$




$\leftarrow C(e^{j\omega})$

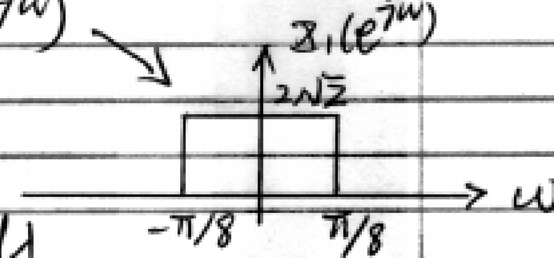



$\leftarrow D(e^{j\omega})$




$\leftarrow E(e^{j\omega}) = X(e^{j\omega})$

5. (a) Define $x_1(n) \leftrightarrow X_1(e^{j\omega})$




Then $X_1(e^{j\omega}) \circ X_1(e^{j\omega})$

$$= \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\lambda})\, X_1(e^{j(\omega-\lambda)})\, d\lambda$$

$$= X(e^{j\omega})$$

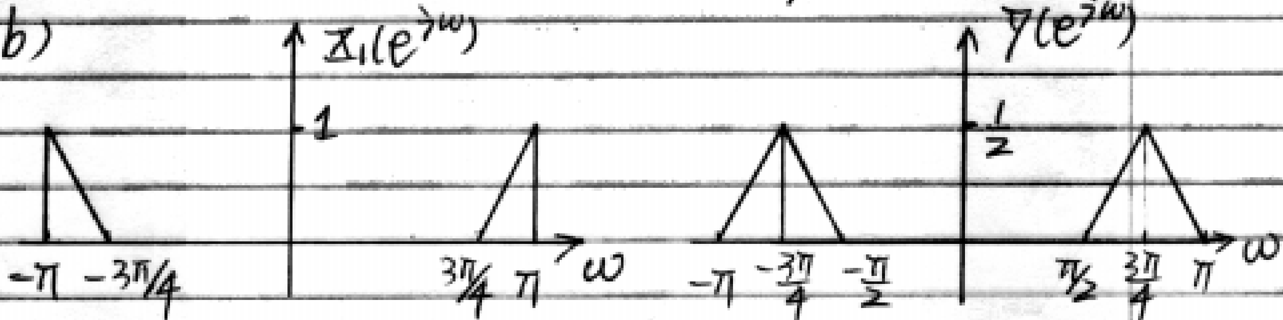
$\therefore x(n) = x_1(n)\, x_1(n) = [x_1(n)]^2$

with $x_1(n) = 2\sqrt{2}\,\dfrac{\text{sinc}(\frac{\pi}{8}n)}{8} = \dfrac{\text{sinc}(\frac{\pi}{8}n)}{2\sqrt{2}}$

$\Rightarrow y(n) = (-1)^n \cos(\frac{\pi}{4}n)\, x(n)$

$$= (-1)^n \cos(\tfrac{\pi}{4}n)\, \text{sinc}^2(\tfrac{\pi}{8}n)/8$$

(b)




(c) $\displaystyle\sum_{n=-\infty}^{\infty} |y(n)|^2 = \frac{1}{2\pi}\int_{2\pi} |Y(e^{j\omega})|^2\, d\omega$

$$Y(e^{j\omega}) = \begin{cases} -\frac{2}{\pi}\omega + 2 & , \omega \in [\frac{3\pi}{4}, \pi] \\ \frac{2}{\pi}\omega - 1 & , \omega \in [\frac{\pi}{2}, \frac{3\pi}{4}] \\ -\frac{2}{\pi}\omega - 1 & , \omega \in [-\frac{3\pi}{4}, -\frac{\pi}{2}] \\ \frac{2}{\pi}\omega + 2 & , \omega \in [-\pi, -\frac{3\pi}{4}] \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{1}{2\pi}\int_{2\pi}|Y(e^{j\omega})|^2 d\omega$$

$$=\frac{1}{2\pi}\int_{3\pi/4}^{\pi}\left(-\frac{2}{\pi}\omega+2\right)^2 d\omega+\frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\left(\frac{2}{\pi}\omega-1\right)^2 d\omega$$

$$+\frac{1}{2\pi}\int_{-\frac{3\pi}{4}}^{-\frac{\pi}{2}}\left(-\frac{2}{\pi}\omega-1\right)^2 d\omega+\frac{1}{2\pi}\int_{-\pi}^{-\frac{3\pi}{4}}\left(\frac{2}{\pi}\omega+2\right)^2 d\omega$$

consider one of four integrations

$$\frac{1}{2\pi}\int_{\pi/2}^{3\pi/4}\left(\frac{2}{\pi}\omega-1\right)^2 d\omega$$

$$=\frac{1}{2\pi}\int_{\pi/2}^{3\pi/4}\left(\frac{2}{\pi}\omega-1\right)^2\cdot\frac{\pi}{2}\,d\left(\frac{2}{\pi}\omega-1\right)$$

define $\lambda \triangleq \left(\frac{2}{\pi}\omega-1\right)$

$$\Rightarrow \frac{1}{4}\int_0^{\frac{1}{2}}\lambda^2 d\lambda=\frac{1}{4}\frac{\lambda^3}{3}\Big|_0^{\frac{1}{2}}=\frac{1}{96}$$

Evaluation of other three integration results in the same numbe $1/96$

$$\therefore E_y=\sum_{n=-\infty}^{\infty}|y(n)|^2=\frac{1}{2\pi}\int_{2\pi}|Y(e^{j\omega})|^2 d\omega$$

$$=4\times\frac{1}{96}$$

$$=\frac{1}{24}$$

(d) The 5-point DFT of $y(n)$ is obtained by sampling $Y(e^{j\omega})$ at $\frac{2\pi k}{5}$ & scale result by $\frac{2\pi}{N}$

$\Rightarrow$ $N=5$

$\Rightarrow$ $Y(k=0) = 0$

$Y(k=1) = 0$

$Y(k=2) = \cancel{Y}\ \frac{2\pi}{5} Y(e^{j\frac{4\pi}{5}}) = \frac{4\pi}{25}$

$Y(k=3) = \frac{2\pi}{5} Y(e^{j\frac{6\pi}{5}}) = \frac{4\pi}{25}$

$Y(k=4) = 0$