Total: 100 points

ECE113: Digital Signal Processing

Midterm 1

12:00 pm - 1:40 pm, Apr 24, 2019

NAME ________ **UID:**

This exam has 3 problems, for a total of 100 points.

Closed book. No calculators. No electronic devices.
One page, letter-size, one-side cheat-sheet allowed.
Answer the questions in the space provided below each problem. If you run out of room for an answer, continue on the back of the page or use the extra pages at the end.
Please, write your name and UID on the top of each loose sheet!
GOOD LUCK!

Problem	Points	Total Points
1	24	24
2	28	40
3	28	36
Total	80	100

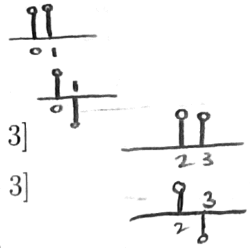
Extra Pages: ________

To fill in, in case extra sheets are used apart from what is provided.

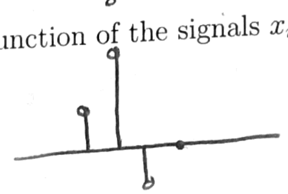
Note: Answers without justification will not be awarded any marks.

Problem 1 (24 points): Consider the four signals:

$$x_1[n] = \delta[n] + \delta[n-1]$$
$$x_2[n] = \delta[n] - \delta[n-1]$$
$$x_3[n] = \delta[n-2] + \delta[n-3]$$
$$x_4[n] = \delta[n-2] - \delta[n-3]$$



(a) **(8 points)** Express the signal $x[n] = \Big\{\underset{\underset{n=0}{\uparrow}}{1},\ 3, -1, 0\Big\}$ as a function of the signals $x_i[n]$.



$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \\ 0 \end{bmatrix}$$

$a + b = 1$ $\qquad 2a = 4$

$a - b = 3$ $\qquad a = 2 \,;\, b = -1$

$c + d = -1$ $\qquad 2c = -1$

$c - d = 0$ $\qquad c = -1/2 \qquad d = -1/2$

8

2	2	0	0
-1	1	0	0
0	0	-1/2	-1/2
0	0	-1/2	1/2
1	3	-1	0

$$\boxed{x[n] = 2x_1[n] - x_2[n] - \tfrac{1}{2}x_3[n] - \tfrac{1}{2}x_4[n]}$$

(b) **(8 points)** Consider again the signals $x_i[n]$ $i = 1, 2, 3, 4$ and $x[n]$ from part (a) of this problem. You are given that the convolution of the signals $x_1[n]$ with $y[n]$ gives the signal $z_1[n]$, that is,

$$z_1[n] = x_1[n] * y[n]$$

You are also given that

$$z_2[n] = x_4[n] * y[n]$$

Calculate the convolution $x[n] * y[n]$ as a function of $z_1[n]$ and $z_2[n]$.

$$x[n] * y[n] = \left(2x_1[n] - x_2[n] - \tfrac{1}{2}x_3[n] - \tfrac{1}{2}x_4[n]\right) * y[n]$$

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$$= 2x_1[n] * y[n] - x_2[n] * y[n] - \tfrac{1}{2}x_3[n] * y[n] - \tfrac{1}{2}x_4[n] * y[n]$$

$$= 2z_1[n] - \underbrace{x_2[n] * y[n]} - \tfrac{1}{2}\underbrace{x_3[n] * y[n]} - \tfrac{1}{2}z_2[n]$$

find x_2 in terms of x_4

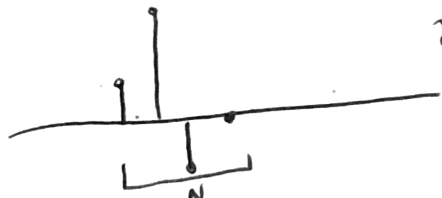
$\Rightarrow x_2[n] = x_4[n+2] \Rightarrow x_4[n+2] * y[n] = z_2[n+2]$

find x_3 in terms of x_1

$x_3[n] = x_1[n-2] \Rightarrow \tfrac{1}{2}x_1[n-2] * y[n] = \tfrac{1}{2}z_1[n-2]$

finally:

$$\boxed{x[n] * y[n] = 2z_1[n] - z_2[n+2] - \tfrac{1}{2}z_1[n-2] - \tfrac{1}{2}z_2[n]}$$



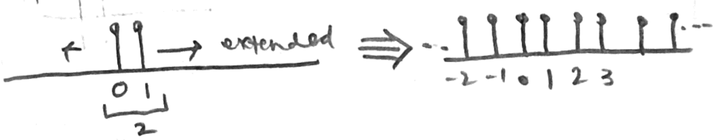
(c) **(8 points)** The periodic extension of a signal $x[n]$ that has length N, is defined to be the signal $x_p[n]$ that simply repeates $x[n]$ every N samples, that is,

$$x_p[n] = \sum_{k=-\infty}^{\infty} x[n-kN]$$

do not shift the original signal

(i) **(4 points)** Consider the periodic extension of the signals $x_1[n]$ and $x_2[n]$ that are given in part (a) of this problem. Are any of these signals even for $N = 2$? If yes which ones?

$x_1[n]$: for $N=2$



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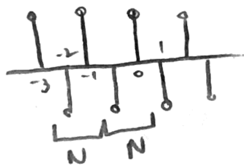
this is even

$x_2[n]$: for $N=2$



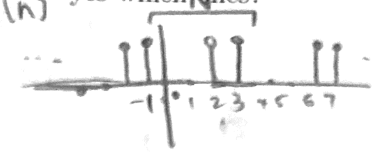
this satisfies the property that $x[n-2] = x[n]$ $\Rightarrow$ NOT even

any $x_1[n]$ is even when it has period of 2.

(ii) **(4 points)** Now consider the periodic extension of the signals $x_3[n]$ and $x_4[n]$ that are given in part (a) of this problem. Are any of these signals even for $N = 4$? If yes which ones?

4

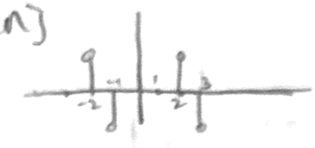
$x_3(n)$



$x[n] = x[n-4]$

$\Rightarrow$ NOT even

$x_4[n]$



$x[n] = x(n-4)$

$\Rightarrow$ NOT even

$\ddagger$

Problem 2 (40 points) : The following questions are not related to each other.

(a) **(8 points)** A one-dimensional linear classifier takes as input a feature value x and outputs 1 if x is larger than a constant A and zero otherwise.

(i) **(4 points)** Is this system linear?

$$y(x) = \begin{cases} 1 & x > A \\ 0 & \text{ow} \end{cases} \Rightarrow \text{shifted unit step}$$

$y[x] = u[x - A]$

$y_1[x] = u_1[x_1 - A]$

$y_2[x] = u_2[x_2 - A]$

$x_3 = ax_1 + bx_2$

$y_3[x] = u[(ax_1 + bx_2) - A] \neq a y_1[x] + b y_2[x]$

NOT linear

4

(ii) **(4 points)** Is the system time invariant?

$$y[x] = \begin{cases} 1 & x > A \\ 0 & \text{otherwise} \end{cases}$$

$y(x)$:

→ unit step $u[x-A]$

shifting input

$u[x-A-k]$

is going to shift your output:

$y[x-k] = u[x-A-k] \Longrightarrow$ Time invariant

4

(b) **(12 points)** Assume that the system $y[n] = f(x[n])$, where $f()$ is some unknown function, is Linear Time Invariant (LTI). Can you determine if the following systems are LTI? Briefly explain why.

(i) **(4 points)** $z[n] = y[n-5] + y[n-2]$

$$z[n] = f(x[n-5]) + f(x[n-2])$$

linear: $z_1[n] = f(x_1[n-5]) + f(x_1[n-2])$

$$z_2[n] = f(x_2[n-5]) + f(x_2[n-2])$$

$$x_3[n] = ax_1[n] + bx_2[n]$$

since $f(x[n])$ systems are LTI

$$z_3[n] = f(ax_1[n-5] + bx_2[n-5]) + f(ax_1[n-2] + bx_2[n-5])$$

$$z_3[n] = f(ax_1[n-5]) + f(bx_2[n-5]) + f(ax_1[n-2]) + f(bx_2[n-5])$$

linearity of systems $f(x)$

$$z_3[n] = af(x_1[n-5]) + af(x_1[n-2]) + bf(x_2[n-5]) + bf(x_2[n-2])$$

recombine again due to linearity

$$= af(x_1[n-5] + x_1[n-2]) + bf(x_2[n-5] + x_2[n-2])$$

$$= az_1[n] + bz_2[n] \Rightarrow \boxed{\text{LINEAR}}$$

time invariant.

Shifted input

$$f(x[n-5-k]) + f(x[n-2-k])$$

$$= y[n-5-k] + y[n-5-k]$$

$$= z[n-k]$$

time invariant

$\Longrightarrow$ system is LTI

4

linear!

(ii) **(4 points)** $z[n] = y^2[n]$

$z[n] = (f(x[n]))^2$

$z_1[n] = (f(x_1[n]))^2$

$z_2[n] = (f(x_2[n]))^2$

$x_3[n] = ax_1[n] + bx_2[n]$

$z_3[n] = (f[ax_1[n] + bx_2[n]])^2$

$= [af[x_1[n]] + bf(x_2[n])]^2$

$\neq az_1[n] + bz_2[n]$ because of the square

$\Rightarrow$ NOT linear

time-in

shifted input:

$f(x[n-k])^2$

$= y^2[n-k]$

$= z[n-k]$

$\Rightarrow$ time invariant

4

NOT LTI

(iii) **(4 points)** $z[n] = y[2n]$

$z[n] = f(x[2n])$

time inv:

shift input

$f(x[2n-k])$

$= f(x[2(n-k/2)])$

$= y[2n-k]$

$= z[n-k/2]$

$\neq z[n-k] \Rightarrow$ Not time invariant

linear

$z_1[n] = f(x_1[2n])$

$z_2[n] = f(x_2[2n])$

$x_3[n] = ax_1[n] + bx_2[n]$

$z_3[n] = f(ax_1[2n] + bx_2[2n])$

$= af(x_1[2n]) + bf(x_1[2n])$ ← b/c f is LTI

$= az_1[n] + bz_2[n]$

$\Rightarrow$ linear

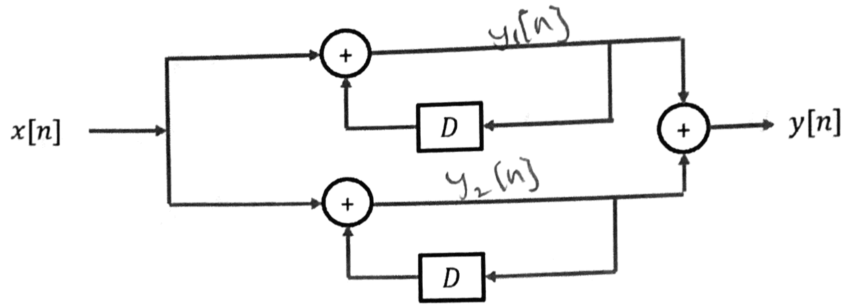


Figure 1: System for Problem 2(c)

(c) **(20 points)** Consider a relaxed system as shown in Fig. 1

(i) **(8 points)** Can you write the input-output equations for this system?

zero before n=0 → their sum is zero

$$y[n] = y_1[n] + y_2[n]$$

$$\begin{cases} y_1[n] = x[n] + y_1[n-1] \\ y_2[n] = x[n] + y_2[n-1] \end{cases}$$

$$\boxed{y[n] = 2\sum_{k=0}^{n} x[k]}$$

6

$$y_1[0] = x(0) + y_1(-1)$$

$$y_1(0) = x(0)$$

$$y_1[1] = x[1] + y_1[0]$$

$$= x[1] + x(0)$$

$$y_1(2) = x[2] + y[1]$$

$$= x(2) + x[1] + x[0)$$

$$y_1(n) = \sum_{k=0}^{n} x[k]$$

likewise,

$$y_2(n) = \sum_{k=0}^{n} x[k]$$

#

(ii) **(4 points)** Is this a BIBO stable system? (explain why?)

$$y[n] = 2 \sum_{k=0}^{n} x[k]$$

let us bound $|x[n]| < B$

$$y[n] \leq 2 \sum_{k=0}^{n} B$$

however, if $n \to \infty$, this summation is infinite

$\Longrightarrow$ NOT stable!

4

(iii) **(4 points)** Assume you connect two of these causal systems, shown in Fig. 1, in series. What is the impulse response of the overall equivalent system?

$y(n) = 4\sum_{k=0}^{n} x(k)$ is the output from the causal system

$y_{new} = 4\sum_{j=0}^{n} y(j) = 4\sum_{j=0}^{n}\sum_{k=0}^{j} x(k)$

impulse response → $x(n) = \delta(n)$

$= 4\sum_{j=0}^{n}\sum_{k=0}^{j} \delta(k)$

$= 4\sum_{j=0}^{n} 1$

$= 4(\underbrace{1+1+\cdots}_{n+1})$

$h(n) = 4n$

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(iii) **(8 points)** Assume you connect two of these relaxed systems, shown in Fig. 1, in series, what is the impulse response of the overall equivalent system?

$y[n] = 2\sum_{k=0}^{n} x[k]$ is the input for the new system

$y_{new} = 2\sum_{j=0}^{n} y[j] = 4\sum_{j=0}^{n}\sum_{k=0}^{n} x[k]$

impulse response → $x(n) = \delta(n)$

$4\sum_{j=0}^{n}\sum_{k=0}^{n} \delta[k]$

$= 4\sum_{j=0}^{n} 1$

$= 4(\underbrace{1+1+1+1}_{n \text{ times}})$

$h(n) = 4n$

0

Problem 3 (36 points) : The following questions are not related to each other.

$\nearrow c_k^* = c_{-k} = c_{N-k}$

(a) **(14 points)** True or false: Let $x[n]$ be a real periodic signal with even period N, and c_k be the coefficients of its associated discrete-time Fourier series. Then, there always exist j_1, j_2 such that c_{j_1} and c_{j_2} are purely real. Mathematically justify your answer.

$$x[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi k}{N} n} \quad \rightarrow N \text{ is even}$$

$$= \left(\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} c_k e^{j 2\pi n \frac{k}{N}} \right)^*$$

$$c_k^* = c_{-k} = c_{N-k}$$

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} c_k^* e^{-j\frac{2\pi n k}{N}}$$

ran out of time

→ see other

6.

(b) **(8 points)** Find the discrete-time Fourier Series coefficients for the following signal:

$$x[n] = \sin(\frac{2\pi n}{3})\cos(\frac{\pi n}{3})$$

$$x[n] = \left(\frac{e^{j\frac{2\pi}{3}n} - e^{-j\frac{2\pi}{3}n}}{2j}\right) \cdot \left(\frac{e^{j\frac{\pi}{3}n} + e^{-j\frac{\pi}{3}n}}{2}\right)$$

the overall period is going to be period of sin() times period of cosine

N_1 (of sin)

$$\frac{2\pi}{3} = 2\pi f_1 \qquad\qquad \frac{\pi}{3} = 2\pi f_2$$

$$f_1 = \frac{1}{3} \qquad\qquad f_2 = \frac{1}{6}$$

$$N = \frac{k}{f_1} = 3k ; \boxed{N_1 = 3} \text{ for } k=1 \qquad\qquad N = \frac{k}{f_2} = 6k ; \boxed{N_2 = 6}$$

overall period is $\boxed{N = 6}$

$$\frac{1}{4j}\left(\cancel{e^{j\pi n}} - e^{-j\frac{\pi}{3}n} + e^{j\frac{\pi}{3}n} - \cancel{e^{-j\pi n}}\right)$$

$e^{j\pi n} = -1 \quad \cos(\pi) + j\sin(\pi)$

$e^{-j\pi n} = -1 \quad \cos(\pi) - j\sin(-\pi)$

$$\frac{1}{4j}\left(-e^{-\frac{2\pi}{6}n} + e^{j\frac{2\pi}{6}n}\right)$$

$$\boxed{c_k = \begin{cases} -\frac{1}{4j}, & k = -1+6 = 5 \\ \frac{1}{4j}, & k = 1 \\ 0, & k \neq -1, 1 \end{cases}}$$

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(c) **(14 points)** Assume that for a periodic signal $x[n]$ with period N you find the DTFS coefficients c_k. As we discussed in class, the Fourier Series coefficients c_k can also be thought as the values of a periodic signal with the same period N. Lets call this periodic signal $c[k]$, that is,

$$c[k] = c_k$$

Your friend claims that if you take the Fourier Series expansion of this periodic signal $c[k]$, you will get coefficients d_m that will be sufficient to retrieve the original signal $x[m]$. Are they right? Mathematically justify your answer.

$$x[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi k}{N}n} \Rightarrow c_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n}$$

FS expansion of $c[k]$ →

$$c[k] = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} d_m e^{j\frac{2\pi m}{N}k} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n}$$

coeffs of $c[k]$ →

$$d_m = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} c[k] e^{-j\frac{2\pi k}{N}m} \checkmark$$

⇒ change of variables $-k' = k$

$$d_m = \frac{1}{\sqrt{N}} \sum_{k'=0}^{1-N} c[-k'] e^{j\frac{2\pi}{N}k'm}$$

$$d_m = \frac{1}{\sqrt{N}} \sum_{k'=-(N-1)}^{0} c(-k') e^{j\frac{2\pi}{N}k'm}$$ ⇒ still summing over a period, so we can move the sum to a diff. period.

$$d_m = \frac{1}{\sqrt{N}} \sum_{k'=0}^{N-1} c(-k') e^{j\frac{2\pi}{N}k'm}$$

$$d_{-m} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} c(k) e^{-j\frac{2\pi k}{N}m}$$

14 $\boxed{d_m = x[m]}$ → yes we can retrieve the signal!

How? Please justify as well.