Problem 1 (24 points): Consider the four signals:

$$
\begin{aligned}
x_1[n] &= \delta[n] + \delta[n-1] \\
x_2[n] &= \delta[n] - \delta[n-1] \\
x_3[n] &= \delta[n-2] + \delta[n-3] \\
x_4[n] &= \delta[n-2] - \delta[n-3]
\end{aligned}
$$

(a) **(8 points)** Express the signal $x[n] = \left\{ \underset{\underset{n=0}{\uparrow}}{1}, 3, -1, 0 \right\}$ as a function of the signals $x_i[n]$.

$x[n] = \delta[n] + 3\delta[n-1] - \delta[n-2]$

Since $\delta[n] = \dfrac{(x_1[n] + x_2[n])}{2}$,

$\delta[n-1] = \dfrac{x_1[n] - x_2[n]}{2}$

$\delta[n-2] = \dfrac{x_3[n] + x_4[n]}{2}$

$\Rightarrow x[n] = \dfrac{x_1[n] + x_2[n]}{2} + \dfrac{3}{2}(x_1[n] - x_2[n]) - \dfrac{x_3[n] + x_4[n]}{2}$

$= \dfrac{1}{2}(4x_1[n] - 2x_2[n] - x_3[n] - x_4[n])$

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(b) **(8 points)** Consider again the signals $x_i[n]$ $i = 1, 2, 3, 4$ and $x[n]$ from part (a) of this problem. You are given that the convolution of the signals $x_1[n]$ with $y[n]$ gives the signal $z_1[n]$, that is,

$$z_1[n] = x_1[n] * y[n]$$

You are also given that

$$z_2[n] = x_4[n] * y[n]$$

Calculate the convolution $x[n] * y[n]$ as a function of $z_1[n]$ and $z_2[n]$.

$$\cancel{z_1[n] = x_1[n] * y[n]}$$
$$\cancel{= (\delta[n] + \delta[n-1]) * y[n] = y[0] + y[n-1]}$$
$$\cancel{z_2[n] = x_4[n] * y[n]}$$
$$\cancel{= (\delta[n-2] - \delta[n-3]) * y[n] = y[n-2] - y[n-3].}$$
$$\cancel{\text{Therefore, } x[n] * y[n] = (\delta[n] + 3\delta[n-1] - \delta[n-2]) * y[n]}$$
$$\cancel{= y[0] + 3y[1] - y[2]}$$

Since $z_1[n] = x_1[n] * y[n]$, and $x_3[n] = x_1[n-2]$.

$\Rightarrow x_3[n] * y[n] = z_1[n-2]$;

Since $z_2[n] = x_4[n] * y[n]$, and $x_2[n] = x_4[n+2]$

$\Rightarrow x_2[n] * y[n] = z_2[n+2]$.

Therefore, $x[n] * y[n] = \left(2x_1[n] - x_2[n] - \frac{1}{2}x_3[n] - \frac{1}{2}x_4[n]\right) * y[n]$

$$= 2z_1[n] - z_2[n+2] - \frac{1}{2}z_1[n-2] - \frac{1}{2}z_2[n].$$

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(c) **(8 points)** The periodic extension of a signal $x[n]$ that has length N, is defined to be the signal $x_p[n]$ that simply repeates $x[n]$ every N samples, that is,

$$x_p[n] = \sum_{k=-\infty}^{\infty} x[n - kN]$$

(i) **(4 points)** Consider the periodic extension of the signals $x_1[n]$ and $x_2[n]$ that are given in part (a) of this problem. Are any of these signals even for $N = 2$? If yes which ones?

$x_1[n] = \delta[n] + \delta[n-1]$

$x_2[n] = \delta[n] - \delta[n-1]$

Both signal x_1 and x_2 have length of 2 ($n=0$ and $n=1$)

After periodic extension,

$Xp_1[n] = \sum_{k=-\infty}^{\infty} (\delta[n-2k] + \delta[n-2k-1])$,

meaning $Xp_1[n] = 1$ for all $n \in \mathbb{Z}$.

$Xp_2[n] = \sum_{k=-\infty}^{\infty} (\delta[n-2k] - \delta[n-2k-1])$

meaning $Xp_2[n] = \begin{cases} 1 & \text{for } n \text{ is even} \\ -1 & \text{for } n \text{ is odd.} \end{cases}$

Since $Xp_1[-n] = Xp_1[n] = 1$, $Xp_1[n]$ is even;

$Xp_2[-n] \doteq Xp_2[n] = \begin{cases} 1 & \text{for } n \text{ is even} \\ -1 & \text{for } n \text{ is odd} \end{cases}$

Therefore, both Xp_1 and Xp_2 are even

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(ii) **(4 points)** Now consider the periodic extension of the signals $x_3[n]$ and $x_4[n]$ that are given in part (a) of this problem. Are any of these signals even for $N = 4$? If yes which ones?

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$x_3[n] = \delta[n-2] + \delta[n-3]$

$x_4[n] = \delta[n-2] - \delta[n-3]$.

$\Rightarrow x_{p3}[n] = \sum_{k=-\infty}^{\infty} \left(\delta[n-2-4k] + \delta[n-3-4k]\right)$.

$x_{p4}[n] = \sum_{k=-\infty}^{\infty} \left(\delta[n-2-4k] - \delta[n-3-4k]\right)$.

Since $x_{p3}[-1] = 1 \neq x_{p3}[1] = 0$,

$x_{p3}[n]$ is not even;

$\rightarrow$ counter examples

Since $x_{p4}[-1] = -1 \neq x_{p4}[1] = 0$,

$x_{p4}[n]$ is not even.

Problem 2 (40 points) : The following questions are not related to each other.

(a) **(8 points)** A one-dimensional linear classifier takes as input a feature value x and outputs 1 if x is larger than a constant A and zero otherwise.

(i) **(4 points)** Is this system linear?

$$y[n] = \begin{cases} 1, & \text{if } x[n] > A \\ 0, & \text{if } x[n] \leq A. \end{cases}$$

Consider $X_1[n] < A$, $X_2[n] < A$, but $X_1[n] + X_2[n] > A$:

$\Rightarrow Sys\{X_1[n]\} + Sys\{X_2[n]\} = 0 + 0 = 0$

$Sys\{X_1[n] + X_2[n]\} = 1 \neq 0$.

Therefe, system is not linear.

Cauter example.

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(ii) **(4 points)** Is the system time invariant?

$$y[n] = \begin{cases} 1, & \text{if } x[n] > A \\ 0, & \text{if } x[n] \le A. \end{cases}$$

Delayed input:

$$Sys\{x[n-m]\} = \begin{cases} 1, & \text{if } x[n-m] > A \\ 0, & \text{if } x[n-m] \le A \end{cases} = y[n-m].$$

Therefore, the system is T.I.

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(b) (12 points) Assume that the system $y[n] = f(x[n])$, where $f()$ is some unknown function, is Linear Time Invariant (LTI). Can you determine if the following systems are LTI? Briefly explain why.

(i) (4 points) $z[n] = y[n-5] + y[n-2]$

$z[n] = y[n-5] + y[n-2]$

$= f(x[n-5]) + f(x[n-2])$

Since it is a linear combination of two LTI system. $z[n]$ is also LTI.

Specifically,

Linearity: $Sys\{a y_1[n] + b y_2[n]\}$

$= a\, sys\{y_1[n]\} + b\, sys\{y_2[n]\}$.

TI: delayed input: $y[n-5-m] + y[n-2-m]$.

delayed output: $z[n-m] = y[n-5-m] + y[n-2-m]$

$\Rightarrow$ LTI.

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(ii) **(4 points)** $z[n] = y^2[n]$

$$z[n] = y^2[n] = f^2(x[n])$$

① Linearity:

① $Sys\{ax_1[n] + bx_2[n]\} = f^2(ax_1[n] + bx_2[n])$

$= (ay_1[n] + by_2[n])^2$

② $Sys\{ax_1[n]\} + Sys\{bx_2[n]\} = (ay_1[n])^2 + (by_2[n])^2$

Since ① ≠ ② ⇒ $z[n]$ is not linear

② TI:

delayed input: $y^2[n-m] = f^2(x[n-m])$

delayed output: $z[n-m] = y^2[n-m] = f^2(x[n-m])$

Therefore, $z[n]$ is TI.

Overall, $z[n]$ is not LTI.

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(iii) **(4 points)** $z[n] = y[2n]$

$z[n] = y[2n] = f(x[2n])$

① Linearity:

$Sys\{ax_1[n] + bx_2[n]\} = f(ax_1[2n] + bx_2[2n]) = ay_1[2n] + by_2[2n]$

$= Sys\{ax_1[n]\} + Sys\{bx_2[n]\} = ay_1[2n] + by_2[2n]$.

$\Rightarrow$ $z[n]$ system is linear;

② TI:

delayed input: $y[2n-m]$.

delayed output: $z[n-m] = y[2(n-m)] = y[2n-2m] \neq y[2n-m]$

Therefore, system $z[n]$ is not TI.

Overall, $z[n]$ system not LTI.

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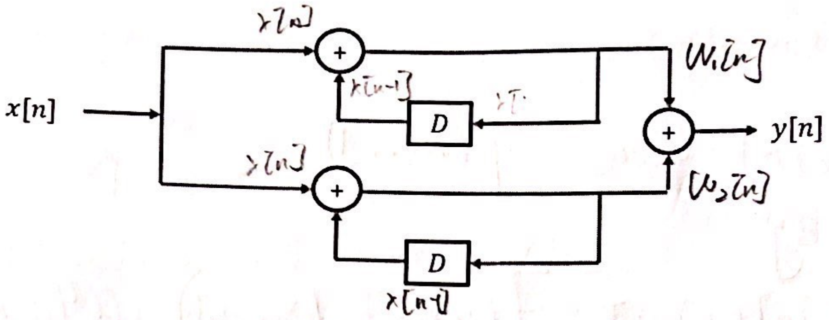

Figure 1: System for Problem 2(c)

(c) **(20 points)** Consider a relaxed system as shown in Fig. 1

(i) **(8 points)** Can you write the input-output equations for this system?

$$y[n] = W_1[n] + W_2[n]$$

$$= (x[n] + x[n-1]) + (x[n] + x[n-1])$$

$$= 2(x[n] + x[n-1])$$

$y[n] = 2x[n] + y[n-1]$

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(ii) **(4 points)** Is this a BIBO stable system? (explain why?)

Since system relaxed:

$x[-\infty]=0$.

Therefore, for both $W_1[n]$ and $W_2[n]$, they act like accumulators:

$$W_1[n]=W_2[n]=\sum_{k=-\infty}^{\infty} x[n-k]$$

While $x[n]$ is bounded, k is unbounded

$\Rightarrow$ both $W_1[n]$ and $W_2[n]$ are unbounded.

Therefore, $y[n]=W_1[n]+W_2[n]$ is not bounded.

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(iii) (8 points) Assume you connect two of these relaxed systems, shown in Fig. 1, in series, what is the impulse response of the overall equivalent system?

$$y[n] = 2(x[n] + x[n-1])$$

$$\Rightarrow h[n] = 2(\delta[n] + \delta[n-1])$$

$$\Rightarrow h_{eq}[n] = h[n] * h[n]$$

$$= (2\delta[n] + 2\delta[n-1]) * (2\delta[n] + 2\delta[n-1])$$

$$= 4\delta[n] + 4\delta[n-1] + 4\delta[n-1] + 4\delta[n-2]$$

$$= 4\delta[n] + 8\delta[n-1] + 4\delta[n-2].$$

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Problem 3 (36 points) : The following questions are not related to each other.

(a) **(14 points)** True or false: Let $x[n]$ be a real periodic signal with even period N, and c_k be the coefficients of its associated discrete-time Fourier series. Then, there always exist j_1, j_2 such that c_{j_1} and c_{j_2} are purely real. Mathematically justify your answer.

$x[n]$: real $\Rightarrow$ C_k is conjugate symmetric

$\Rightarrow x^*[n] = x[n]$, $C_k^* = C_{-k} = C_{N-k}$.

where $C_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N} n}$

Consider $k=0$:

$$C_0 = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{0} = \frac{1}{N} \sum_{n=0}^{N-1} x[n].$$

Since $x[n]$ is real, C_0 is real.

Since $C_k^* = C_{-k} = C_{N-k}$.

We always have C_{aN} being real.

with $a \in \mathbb{Z}$.

Therefore, it is true.

C_{j_1} and C_{j_2} are purely real

when j_1 and j_2 are multiples of N.

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j_1 & j_2 can be in the same period & still C_{j_1} & C_{j_2} will be real.

(b) (8 points) Find the discrete-time Fourier Series coefficients for the following signal:

$$x[n] = \sin(\frac{2\pi n}{3})\cos(\frac{\pi n}{3})$$

$x[n] = \sin(\frac{2\pi n}{3})\cos(\frac{\pi n}{3})$

$= \frac{1}{4}(e^{j\frac{2\pi}{3}n} - e^{-j\frac{2\pi}{3}n})(e^{j\frac{\pi}{3}n} + e^{-j\frac{\pi}{3}n})$

$= \frac{1}{4}(e^{j\pi n} + e^{j\frac{\pi}{3}n} - e^{-j\frac{\pi}{3}n} - e^{-j\pi n})$

$= \frac{1}{4}(0 + e^{j\frac{\pi}{3}n} - e^{-j\frac{\pi}{3}n})$

Since: $\sin(\frac{2\pi}{3}n)$ has period $N_1 = 3$,

$\cos(\frac{\pi n}{3})$ has period $N_2 = 6$

$\Rightarrow$ $x[n]$ has period of 6.

Since $x[n] = \sum_{k=0}^{5} C_k e^{j\frac{\pi}{3}kn}$

$\Rightarrow$ $C_0 = 0$ $\quad C_1 = \frac{1}{4}$

$C_2 = 0$ $\quad C_3 = 0$

$C_4 = 0$ $\quad C_5 = C_{-1} = -\frac{1}{4}$

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(c) **(14 points)** Assume that for a periodic signal $x[n]$ with period N you find the DTFS coefficients c_k. As we discussed in class, the Fourier Series coefficients c_k can also be thought as the values of a periodic signal with the same period N. Lets call this periodic signal $c[k]$, that is,

$$c[k] = c_k$$

Your friend claims that if you take the Fourier Series expansion of this periodic signal $c[k]$, you will get coefficients d_m that will be sufficient to retrieve the original signal $x[m]$. Are they right? Mathematically justify your answer.

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N} n}$$

$$\Rightarrow d_m = \frac{1}{N} \sum_{k=0}^{N-1} c[k] e^{-j\frac{2\pi m}{N} k} = d[m].$$

Since $x[n] = \sum_{k=0}^{N-1} C_k e^{j\frac{2\pi k}{N} n}$,

$$N d[m] = \sum_{k=0}^{N-1} c[k] e^{-j\frac{2\pi k}{N} m}$$

we can see that $x[n] = N d[-m]$.

Therefore, $x[m]$ has values of $N d_{-m}$.

The friend is right.

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