Total: 100 points

ECE113: Digital Signal Processing

Midterm 2

10:00 am - 11:40 am, March 4, 2019

NAME: ______________________________ **UID**: ______________

This exam has 3 problems, for a total of 100 points.

Closed book. No calculators. No electronic devices.
One page, letter-size, two-side cheat-sheet allowed.
Answer the questions in the space provided below each problem. If you run out of room for an answer, continue on the back of the page or use the extra pages at the end.
Please, write your name and UID on the top of each loose sheet!
GOOD LUCK!

Problem	Points	Total Points
1		30
2		30
3		40
Total		100

Extra Pages: ________

To fill in, in case extra sheets are used apart from what is provided.

Note: Answers without justification will not be awarded any marks.

Problem 1 (30 points): The following questions are not related to each other. Please answer the questions within the space provided.

1. (5 points) The signal $x[n]$ is 1 for $n = 0$, and is equal to $\frac{\sin(\pi n)}{n}$ for $n \neq 0$. **True or False:** $x[n]$ is even.

   Solution: True. $x[-n] = x[n]$ since $\sin(-n) = -\sin(n)$.

2. (5 points) Consider a function $X(\Omega) = \cos(0.3\Omega) \quad \forall\, \Omega \in (-\infty, \infty)$. Can this function be the DTFT of a discrete signal? Justify why/why not in a single line.

   Solution: No, since DTFT is periodic with period 2π and this function is not.

3. (7 points) Consider a Linear Time Invariant (LTI) system. Assume that, if the input to this system is the unit step signal $u[n]$, where

   $$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

   the output of the system is a signal $s[n]$. Express the frequency response of the system $H(\Omega)$, as a function of $S(\Omega)$.

   Solution: Impulse response $h[n] = s[n] - s[n-1]$. Therefore $H(\Omega) = S(\Omega) - e^{-j\Omega}S(\Omega)$.

4. (8 points) Find the impulse response $h[n]$ of a system characterized by the input-output equations $y[n] - ay[n-1] = ax[n]$, where $|a| < 1$.

   Solution: The frequency response of the system is $H(\Omega) = \frac{a}{1-ae^{-j\Omega}}$. Taking IDTFT gives $h[n] = a.a^n u[n] = a^{n+1}u[n]$.

5. (5 points) Find the power of $x[n] = \cos(-0.2\pi n - 0.4\pi) \circledast \sum_{l=-\infty}^{\infty} \delta[n + 2 - 10l]$, where $\circledast$ denotes periodic convolution.

   Solution: $x[n] = \cos(-0.2\pi(n+2)) \circledast \sum_{l=-\infty}^{\infty} \delta[n + 2 - 10l]$.
   Period of each signal is 10.
   $x[n] = \cos(-0.2\pi(n+4))$ (convolving with periodic extension of delta shifts it).
   $x[n] = \cos(-2\pi/10(n+4))$.
   Parseval's theorem: power $= \sum_k |c_k|^2$.
   $c_1 = 0.5e^{j(2\pi/10)4}$ and $c_9 = 0.5e^{j(2\pi/10)6}$. All other c_ks are zero (recall the spectrum of cos).

   So Power $= (0.5^2 + 0.5^2) = 2 * 0.5^2 = 0.5$.

Problem 2 (30 points): Consider a periodic signal $\tilde{x}[n] = \{\underline{1}, 2, 3, 4\}$ with period $N = 4$. Here the first sample in $\{\underline{1}, 2, 3, 4\}$ corresponds to $x[0]$, the second corresponds to $x[1]$ and so on. $x[n]$ can be represented using the natural basis functions $\phi_k[n]$ which are also periodic with the same period $N = 4$, as follows:

$$\tilde{x}[n] = \sum_{k=0}^{N-1} c_k \phi_k[n],$$

where we denote by c_k the signal coefficients with respect to this set of basis $\phi_k[n]$, and where $\phi_k[n]$ take values within a period

$$\phi_0[n] = \{\underline{1}, 0, 0, 0\}$$

$$\phi_1[n] = \{\underline{0}, 1, 0, 0\}$$

$$\phi_2[n] = \{\underline{0}, 0, 1, 0\}$$

and

$$\phi_3[n] = \{\underline{0}, 0, 0, 1\}.$$

1. (15 points) Write what the DTFS basis within a period is, and calculate the DTFS coefficients d_k for the signal $\tilde{x}[n]$.

   Solution: DTFS bases are

$$\psi_0[n] = \{e^{2\frac{\pi}{4}0.0}, e^{2\frac{\pi}{4}0.1}, e^{2\frac{\pi}{4}0.2}, e^{2\frac{\pi}{4}0.3}\} = \{1, 1, 1, 1\}$$

$$\psi_1[n] = \{e^{2\frac{\pi}{4}1.0}, e^{2\frac{\pi}{4}1.1}, e^{2\frac{\pi}{4}1.2}, e^{2\frac{\pi}{4}1.3}\} = \{1, j, -1, -j\}$$

$$\psi_2[n] = \{e^{2\frac{\pi}{4}2.0}, e^{2\frac{\pi}{4}2.1}, e^{2\frac{\pi}{4}2.2}, e^{2\frac{\pi}{4}2.3}\} = \{1, -1, 1, -1\}$$

$$\psi_2[n] = \{e^{2\frac{\pi}{4}3.0}, e^{2\frac{\pi}{4}3.1}, e^{2\frac{\pi}{4}3.2}, e^{2\frac{\pi}{4}3.3}\} = \{1, -j, -1, j\}$$

$$d_k = \frac{1}{4} \sum_{n=0}^{4} x[n] e^{-j2\frac{\pi}{4}kn}$$

$$d_0 = (1 + 2 + 3 + 4)/4 = 10/4$$

$$d_1 = (1 + 2j - 3 - 4j)/4 = (-2 - 2j)/4$$

$$d_2 = (1 - 2 + 3 - 4)/4 = -2/4$$

$$d_3 = (1 - 2j - 3 + 4j)/4 = (-2 + 2j)/4$$

2. (15 points) Consider a different basis:

$$\phi'_0[n] = \{\underline{1}, 1, 0, 0\}$$

$$\phi'_1[n] = \{\underline{1}, -1, 0, 0\}$$

$$\phi'_2[n] = \{\underline{0}, 0, 1, 1\}$$

and

$$\phi'_3[n] = \{\underline{0}, 0, 1, -1\}.$$

Let the representation of $x[n]$ in this basis be

$$x[n] = \sum_{k=0}^{3} e_k \phi'_k[n].$$

Find the coefficients e_k of $x[n]$ corresponding to these new basis functions $\phi'_k[n]$.

Solution: One could solve two sets of linear equations in two variables to get $e_0 = 1.5, e_1 = -0.5, e_2 = 3.5, e_3 = -0.5$.
Alternatively, $\phi'_k[n]$ form a set of orthogonal basis vectors and hence the coefficients can be determined as $e_k = \langle x[n], \phi'_k[n]\rangle / ||\phi'_k[n]||^2$. Note that $||\phi'_k[n]||^2 = 2$.
Using this:

$$e_0 = \langle x[n], \phi'_0[n]\rangle/2 = (1+2)/2$$

$$e_1 = \langle x[n], \phi'_1[n]\rangle/2 = (1-2)/2$$

$$e_2 = \langle x[n], \phi'_2[n]\rangle/2 = (3+4)/2$$

$$e_3 = \langle x[n], \phi'_3[n]\rangle/2 = (3-4)/2$$

Problem 3 (40 points): Assume you are given the DTFT $X(\Omega)$ of a signal $x[n]$ that starts at 0 and has length N_1. You sample $X(\Omega)$ at N_2 points (as you need to store it digitally), and keep these N_2 samples $X(\frac{2\pi k}{N_2})$, $k = 0, \ldots, N_2 - 1$. Unfortunately because you did not know what N_1 was, you used $N_2 < N_1$. You now try to apply IDFT to recover your original signal $x[n]$.

(a) (25 points) Show that $X(\frac{2\pi k}{N_2})$ is the DFT of a signal $y[n]$ (of length N_2), whose periodic extension is $y_{ps}[n] = \sum_{\ell=-\infty}^{+\infty} x[n - \ell N_2]$. That is, $y[n]$ is an "aliased in time" version of $x[n]$.

(b) (15 points) Assume that $x[n] = a^n u[n]$, $0 < a < 1$, and thus $X(\Omega) = \frac{1}{1-ae^{-i\Omega}}$. Assume that we take N samples of $X(\Omega)$, what will $y_{ps}[n]$ be?

You may find the infinite geometric summation formula useful:

$$\sum_{i=-\infty}^{a} r^{-i} = \frac{r^{-a}}{1-r}, \quad \text{for} \quad |r| < 1.$$

Solution:

(a) First

$$X(\frac{2\pi k}{N_2}) = X(\Omega)|_{\Omega=\frac{2\pi k}{N_2}} = \sum_{n=-\infty}^{\infty} x[n]e^{-i2\pi kn/N_2}$$

Let $Y[k]$ is the DFT of $y[n]$. Note that in the interval $\{0, 1, ..., N_2 - 1\}$, $y_{ps}[n] = y[n]$. Therefore,

$$\begin{aligned} Y[k] &= \sum_{n=0}^{N_2-1} y[n]e^{-i2\pi kn/N_2} \\ &= \sum_{n=0}^{N_2-1} y_{ps}[n]e^{-i2\pi kn/N_2} \\ &= \sum_{n=0}^{N_2-1} \sum_{l=-\infty}^{\infty} x[n - lN_2]e^{-i2\pi kn/N_2} \end{aligned}$$

Now, let $m = n - lN_2$:

$$\begin{aligned} Y[k] &= \sum_{l=-\infty}^{\infty} \sum_{m=-lN_2}^{-lN_2+N_2-1} x[m]e^{-i2\pi km/N_2} \\ &= \sum_{m=-\infty}^{\infty} x[m]e^{-i2\pi km/N_2} \\ &= X(\Omega)|_{\Omega=\frac{2\pi k}{N_2}} = X(\frac{2\pi k}{N_2}) \end{aligned}$$

(b)

$$\begin{aligned}
y_{ps}[n] &= \sum_{l=-\infty}^{\infty} x[n-lN] \\
&= \sum_{l=-\infty}^{\infty} a^{n-lN} u[n-lN] \\
&= a^n \sum_{l=-\infty}^{n/N} a^{-lN} \\
&= a^n \left(\frac{a^{-n}}{1-a^N}\right) = \frac{1}{1-a^N}
\end{aligned}$$