Total: 100 points

# ECE113: Digital Signal Processing

Midterm 2

12:00 pm - 1:40 pm, May 20, 2019

**NAME**: ______________________ **UID**: ______________

This exam has 5 problems, for a total of 100 points.

Closed book. No calculators. No electronic devices.

One page, letter-size, two-side cheat-sheet allowed.

Answer the questions in the space provided below the problems. If you run out of room for an answer, continue on the back of the page or use the extra pages at the end.

The formula sheet for DTFS, DTFT, and DFT is attached at the end of this midterm

**Please, write your name and UID on the top of each loose sheet!**

**GOOD LUCK!**

| Problem | Points | Total Points |
|---|---|---|
| 1 | | 16 |
| 2 | | 12 |
| 3 | | 24 |
| 4 | | 20 |
| 5 | | 28 |
| Total | | 100 |

**Extra Pages**: _________

To fill in, in case extra sheets are used apart from what is provided.

**Note: Answers without justification will not be awarded any marks.**

The following questions are not related.

Problem 1**(16 points)** Consider two causal and stable LTI systems, with respective impulse response $h_1[n]$ and $h_2[n]$. State whether the following statements are True or False. You must also provide a mathematical justification:

(a) **(8 points)** "If we connect the given two systems in parallel, the resulting equivalent LTI system is stable."

Solution : Yes.

Say, $x_1[n]$ and $y_1[n]$ are the input-output pair of the first system with impulse response $h_1[n]$. Similarly, $x_2[n]$ and $y_2[n]$ are the input-output pair of the other system with impulse response $h_2[n]$.

Let $x[n]$ be the input and $y[n]$ be the output of the equivalent parallel system. Then

$$y[n] = y_1[n] + y_2[n]$$

. If $|x[n]| \leq B < \infty$, then $|y_1[n]| < \infty$ and $|y_2[n]| < \infty$

$$|y[n]| = |y_1[n] + y_2[n]| \leq |y_1[n]| + |y_2[n]| < \infty$$

Alternate Solution:

Since the systems are given to be stable, we know that

$$\sum_{n=0}^{\infty} |h_1[n]| < \infty$$
$$\sum_{n=0}^{\infty} |h_2[n]| < \infty$$

Now, the equivalent system is $h[n] = h_1[n] + h_2[n]$ , therefore we have

$$\begin{aligned}\sum_{n=0}^{\infty} |h[n]| &= \sum_{n=0}^{\infty} |h_1[n] + h_2[n]| \\ &\leq \sum_{n=0}^{\infty} |h_1[n]| + \sum_{n=0}^{\infty} |h_2[n]| < \infty\end{aligned}$$

Therefore, the overall system is also stable.

(b) **(8 points)** "If we connect the given two systems in series, the resulting equivalent LTI system is non-causal."

Solution : No.

Say, $x_1[n]$ and $y_1[n]$ are the input-output pair of the first system with impulse response $h_1[n]$. Similarly, $x_2[n]$ and $y_2[n]$ are the input-output pair of the other system with impulse response $h_2[n]$.

Let $x[n]$ be the input and $y[n]$ be the output of the equivalent parallel system. Then

$$x[n] \to w[n] \to y[n]$$

Now, $w[n]$ depends only on the present or the past inputs, $\implies h_1[n]$ is a causal signal. Similarly, $h_2[n]$ is also a causal signal. The impulse response of the overall system will be

$$h[n] = h_1[n] * h_2[n]$$

Therefore, $h[n]$ is also a causal signal implying that the overall system is causal.

Problem 2**(12 points)** State whether the following statement is True or False. You must provide a mathematical justification for you answer:
"Let us consider an LTI system. Then, two distinct inputs (distinct inputs are inputs that differ in at least one position) will always lead to distinct outputs."

Solution : No. Consider a Low pass filter system with system function

$$H(\Omega) = \begin{cases} 1, & \text{if } -\Omega_c \leq \Omega \leq \Omega_c \\ 0, & \text{otherwise} \end{cases}$$

Now consider two input signals $x_1[n]$ and $x_2[n]$ such that their DTFTs are

$$X_1(\Omega) \begin{cases} = X_2(\Omega), & \text{if } -\Omega_c \leq \Omega \leq \Omega_c \\ \neq X_2(\Omega), & \text{otherwise} \end{cases}$$

Then the corresponsing output signals $y_1[n]$ and $y_2[n]$ will be the same.

Problem 3**(24 points)** Consider a signal $x[n]$ that has the DTFT depicted on Fig. 1 (we depict DTFT only for one period in $[-\pi, \pi]$).

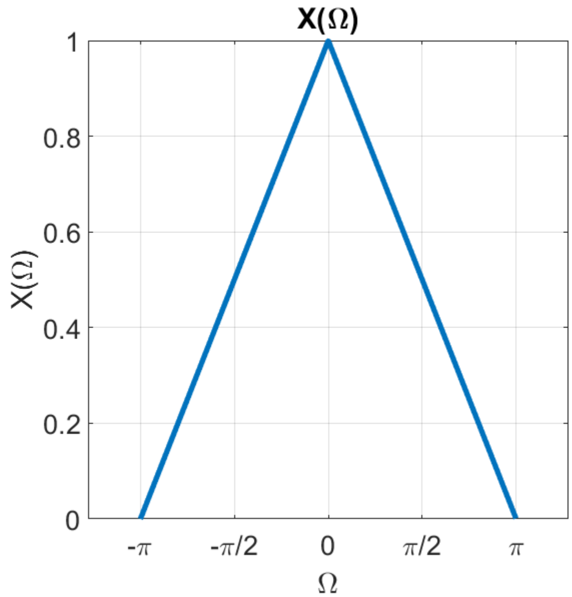


Figure 1: DTFT of $x[n]$

Find the expression for the DTFT of the following signals:

(a) **(8 points)** $x_1[n] = nx[n-1]$.

Solution :

$$X_1\Omega = j\frac{d}{d\Omega}\{e^{-j\Omega}X(\Omega)\}$$

(b) **(8 points)** $x_2[n] = e^{i\frac{\pi n}{2}}x[n] * x[n]$, where $*$ stands for the linear convolution.

Solution :

$$X_2(\Omega) = X(\Omega - \frac{\pi}{2})X(\Omega)$$

(c) **(8 points)** $x_e[n]$, the even part of $x[n]$.

Solution :

$$x_e[n] = \frac{x[n] + x[-n]}{2}$$

Therefore,

$$X_e(\Omega) = \frac{X(\Omega) + X(-\Omega)}{2}$$

Problem 4 **(20 points)** Prove the following expression for any two periodic sequences $x[n]$ and $y[n]$, both with same period $N$ and DTFS coefficients $\{c_k\}$ and $\{d_k\}$, respectively.

$$\sum_{n=0}^{N-1} x[n]y^*[n] = N\sum_{k=0}^{N-1} c_k d_k^*$$

Solution :

$$\begin{aligned}
N\sum_{k=0}^{N-1} c_k d_k^* &= N\sum_{k=0}^{N-1}\left(\frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}\right) d_k^* \\
&= \sum_{n=0}^{N-1} x[n]\left(\sum_{k=0}^{N-1} d_k^* e^{-j2\pi kn/N}\right) \\
&= \sum_{n=0}^{N-1} x[n]\left(\sum_{k=0}^{N-1} d_k e^{j2\pi kn/N}\right)^* \\
&= \sum_{n=0}^{N-1} x[n]y^*[n]
\end{aligned}$$

Since,

$$\sum_{k=0}^{N-1} d_k e^{j2\pi kn/N} = y[n]$$

Problem 5**(28 points)** A permutation is a function that re-orders the elements of a vector, by moving each element i to a position j. Sometimes it is called interleaving. Permutations are bijective (hence invertible, where one can consider that the inverses are also some permutations).

Consider a permutation that maps position $i = 0, \ldots, N-1$ of a vector of length $N$, to the position $[(i\sigma + \tau) \mod N]$ of the same vector, where $\sigma$, and $\tau$ are given constants both taking values in $[0, \cdots, N-1]$. In order to guarantee a one-to-one mapping of the positions after the permutations, we consider $\sigma$ to be coprime with $N$, i.e., the only positive integer that divides both $\sigma$ and $N$ is 1. Moreover, $\sigma$ is invertible modulo $N$, that is, there exists another integer in $[0, \cdots, N-1]$, which we will denote as $\sigma^{-1}$, such that $(\sigma^{-1}\sigma) \mod N = (\sigma\sigma^{-1}) \mod N = 1$, and accordingly,
if

$$(i\sigma + \tau) \mod N = k$$

then

$$i = (k - \tau)\sigma^{-1} \mod N$$

For example, if we permute a signal $x[n] = \{a, b, c, d\}$ of length $N = 4$, using $\tau = 1$ and $\sigma = 3$, we get the signal $y[n] = \{b, a, d, c\}$.
The inverse of $\sigma = 3$ is the number $\sigma^{-1} = 3$, as $\sigma * \sigma^{-1} \mod 4 = 9 \mod 4 = 1$.

Assume that we have a signal $x[n]$ of length $N$, i.e., $x[n]$ has non-zero values for $n = 0, 1, \ldots, N-1$. Now we permute the $i^{th}$ position in $x[n]$ to $[(i\sigma + \tau) \mod N]$ position. Lets say the signal obtained after the position permutation is $y[n]$, which is again of the same length $N$.

Now, consider $X[k]$ and $Y[k]$ to be the DFT of the signals $x[n]$ and $y[n]$, respectively.

(a) **(24 points)** Prove that

$$Y[\sigma i \mod N] = X[i]w^{-\tau i}$$

where $w = e^{-j\frac{2\pi}{N}}$.

Solution : We have,

$$y[n] = x[\sigma n + \tau \mod N]$$

Then the DFT of $y[n]$ is $Y[k]$.

$$\begin{aligned} Y[k] &= \sum_{n=0}^{N-1} y[n] e^{-j\frac{2\pi kn}{N}} \\ &= \sum_{n=0}^{N-1} x[\sigma n + \tau \mod N] e^{-j\frac{2\pi kn}{N}} \end{aligned}$$

Let

$$m = (\sigma n + \tau) \mod N$$

Therefore,

$$n = (m - \tau)\sigma^{-1} \mod N$$

$$\begin{aligned} &= \sum_{n=0}^{N-1} x[m] e^{-\frac{2\pi k((m-\tau)\sigma^{-1})}{N}} \\ &= e^{j\frac{2\pi k(\tau\sigma^{-1})}{N}} \sum_{m=0}^{N-1} x[m] e^{-j\frac{2\pi m(k\sigma^{-1})}{N}} \\ Y[k] &= w^{-\tau(k\sigma^{-1})} \sum_{m=0}^{N-1} x[m] e^{-j\frac{2\pi m(k\sigma^{-1})}{N}} \\ Y[k] &= w^{-\tau(k\sigma^{-1})} X[k\sigma^{-1} \mod N] \\ Y[\sigma k \mod N] &= w^{-\tau k} X[k] \end{aligned}$$

(b) **(4 points)** Show that permuting a vector $x[n]$ in the time domain, results in permuting the magnitude of its DFT $X[k]$.

Solution : Since we have proved

$$Y[\sigma k \mod N] = w^{-\tau k} X[k]$$

Taking the magnitude on both sides gives

$$\begin{aligned} |Y[\sigma k \mod N]| &= |w^{-\tau k}||X[k]| \\ &= |X[k]| \end{aligned}$$

# DTFS and DFT cheat sheet

## DTFS

$\tilde{x}[n]$ is periodic signal with period $N$, and let $\tilde{c}_k$ be the DTFS coefficients of $\tilde{x}[n]$, then:

- Synthesis equation: $\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{c}_k e^{j\frac{2\pi}{N}kn}$
- Analysis equation: $\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn}$

### Properties

1. Periodicity: $\tilde{c}_k$ is periodic with period $N$.
2. Time shift: $\tilde{x}[n-m] \longleftrightarrow e^{-j\frac{2\pi}{N}km} c_k$
3. Symmetry: $\tilde{x}[n]$ purely real implies $\tilde{c}_k^* = \tilde{c}_{N-k}$
   $\tilde{x}[n]$ purely imaginary implies $\tilde{c}_k^* = -\tilde{c}_{N-k}$
   $\tilde{x}[n]$ real and even implies $\tilde{c}_k$ purely real
   $\tilde{x}[n]$ real and odd implies $\tilde{c}_k$ purely imaginary
4. Periodic convolution: If $\tilde{x}[n] \longleftrightarrow \tilde{c}_k$ and $\tilde{y}[n] \longleftrightarrow \tilde{d}_k$, then
$$\tilde{z}[n] = \tilde{x}[n] \circledast \tilde{y}[n] \longleftrightarrow N\tilde{c}_k\tilde{d}_k$$
$$\text{where} \quad z[n] \triangleq \sum_{m=0}^{N-1} x[m]y[n-m]$$
5. Parseval's theorem:
$$\frac{1}{N}\sum_{n=0}^{N-1} |\tilde{x}[n]|^2 = \sum_{k=0}^{N-1} |\tilde{c}_k|^2$$

## DFT

$x[n]$ is a signal with non-zero values only in $\{0, 1, 2, ..., N-1\}$, and let $X[k]$ be the DFT of $x[n]$, then:

- Synthesis equation: $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$
- Analysis equation: $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$
- Relation to DTFT $X[k] = X(\Omega)\big|_{\Omega=\frac{2\pi k}{N}}$

### Properties

1. Time shift: $x[(n-m) \mod N] \longleftrightarrow e^{-j\frac{2\pi}{N}km} X[k]$
2. Frequency shift: $x[n]e^{j\frac{2\pi}{N}mn} \longleftrightarrow X[(k-m) \mod N]$
3. Time reversal: $x[-n \mod N] \longleftrightarrow X[-k \mod N]$
4. Symmetry: $x[n]$ purely real implies $X[k]^* = X[-k \mod N]$
   $x[n]$ purely imaginary implies $X[k]^* = -X[-k \mod N]$
   $x^*[n] = x[-n \mod N]$ implies $X[k]$ purely real
   $x^*[n] = -x[-n \mod N]$ implies $X[k]$ purely imaginary
5. Circular convolution: If $x[n] \longleftrightarrow X[k]$ and $y[n] \longleftrightarrow Y[k]$, then
$$z[n] = x[n] \circledast y[n] \longleftrightarrow X[k]Y[k]$$
$$\text{where} \quad z[n] \triangleq \sum_{m=0}^{N-1} x[m]y[(n-m) \mod N]$$

# DTFT Cheat Sheet (Courtesy of Signals and Systems by Alkin, page 461-462)

| Name | Signal | Transform |
|---|---|---|
| Discrete-time pulse | $x[n] = \begin{cases} 1\,, & \lvert n\rvert \le L \\ 0 & \text{otherwise} \end{cases}$ | $X(\Omega) = \dfrac{\sin\left(\dfrac{(2L+1)\,\Omega}{2}\right)}{\sin\left(\dfrac{\Omega}{2}\right)}$ |
| Unit-impulse signal | $x[n] = \delta[n]$ | $X(\Omega) = 1$ |
| Constant-amplitude signal | $x[n] = 1\,,\quad \text{all}\ n$ | $X(\Omega) = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - 2\pi m)$ |
| Sinc function | $x[n] = \dfrac{\Omega_c}{\pi}\ \text{sinc}\left(\dfrac{\Omega_c n}{\pi}\right)$ | $X(\Omega) = \begin{cases} 1\,, & \lvert\Omega\rvert < \Omega_c \\ 0\,, & \text{otherwise} \end{cases}$ |
| Right-sided exponential | $x[n] = \alpha^n\, u[n]\,,\quad \lvert\alpha\rvert < 1$ | $X(\Omega) = \dfrac{1}{1 - \alpha\, e^{-j\Omega}}$ |
| Complex exponential | $x[n] = e^{j\Omega_0 n}$ | $X(\Omega) = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi m)$ |

**Table 5.4** – Some DTFT transform pairs.

| Theorem | Signal | Transform | |
|---|---|---|---|
| Linearity | $\alpha\, x_1[n] + \beta\, x_2[n]$ | $\alpha\, X_1(\Omega) + \beta\, X_2(\Omega)$ | |
| Periodicity | $x[n]$ | $X(\Omega) = X(\Omega + 2\pi r)$ | for all integers $r$ |
| Conjugate symmetry | $x[n]$ real | $X^*(\Omega) = X(-\Omega)$ | |
| | | Magnitude: | $\lvert X(-\Omega)\rvert = \lvert X(\Omega)\rvert$ |
| | | Phase: | $\Theta(-\Omega) = -\Theta(\Omega)$ |
| | | Real part: | $X_r(-\Omega) = X_r(\Omega)$ |
| | | Imaginary part: | $X_i(-\Omega) = -X_i(\Omega)$ |
| Conjugate antisymmetry | $x[n]$ imaginary | $X^*(\Omega) = -X(-\Omega)$ | |
| | | Magnitude: | $\lvert X(-\Omega)\rvert = \lvert X(\Omega)\rvert$ |
| | | Phase: | $\Theta(-\Omega) = -\Theta(\Omega) \mp \pi$ |
| | | Real part: | $X_r(-\Omega) = -X_r(\Omega)$ |
| | | Imaginary part: | $X_i(-\Omega) = X_i(\Omega)$ |
| Even signal | $x[n] = x[-n]$ | $\text{Im}\{X(\Omega)\} = 0$ | |
| Odd signal | $x[n] = -x[-n]$ | $\text{Re}\{X(\Omega)\} = 0$ | |
| Time shifting | $x[n-m]$ | $X(\Omega)\, e^{-j\Omega m}$ | |
| Time reversal | $x[-n]$ | $X(-\Omega)$ | |
| Conjugation | $x^*[n]$ | $X^*(-\Omega)$ | |
| Frequency shifting | $x[n]\, e^{j\Omega_0 n}$ | $X(\Omega - \Omega_0)$ | |
| Modulation | $x[n]\, \cos(\Omega_0 n)$ | $\frac{1}{2}\left[X(\Omega - \Omega_0) + X(\Omega + \Omega_0)\right]$ | |
| | $x[n]\, \sin(\Omega_0 n)$ | $\frac{1}{2}\left[X(\Omega - \Omega_0)\, e^{-j\pi/2} + X(\Omega + \Omega_0)\, e^{j\pi/2}\right]$ | |
| Differentiation in frequency | $n^m\, x[n]$ | $j^m \dfrac{d^m}{d\Omega^m}\left[X(\Omega)\right]$ | |
| Convolution | $x_1[n] * x_2[n]$ | $X_1(\Omega)\, X_2(\Omega)$ | |
| Multiplication | $x_1[n]\, x_2[n]$ | $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X_1(\lambda)\, X_2(\Omega - \lambda)\, d\lambda$ | |
| Parseval's theorem | $\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \lvert X(\Omega)\rvert^2\, d\Omega$ | | |

**Table 5.3** – DTFT properties.