Total: 100 points

ECE113: Digital Signal Processing

Midterm 1

10:00 am - 11:40 am, Feb 04, 2019

NAME: [redacted] **UID**: [redacted]

This exam has 3 problems, for a total of 100 points.

Closed book. No calculators. No electronic devices.
One page, letter-size, one-side cheat-sheet allowed.
Answer the questions in the space provided below each problem. If you run out of room for an answer, continue on the back of the page or use the extra pages at the end.
Please, write your name and UID on the top of each loose sheet!
GOOD LUCK!

Problem	Points	Total Points
1	32+8	40
2	20+15	35
3	25	25
Total	100	100

Extra Pages: ________

To fill in, in case extra sheets are used apart from what is provided.

Note: Answers without justification will not be awarded any marks.

Problem 1 (40 points): There are three sub-problems to this problem, which are not related.

(a) **(12 points)** If the signal $x[n]$ is real and odd, are the following signals even or odd? (Justify why)

(i) $y_1[n] = x[n^2]$ ~~this~~ $\boxed{\text{even}}$

(ii) $y_2[n] = x[3n]$ $\boxed{\text{odd}}$

(iii) $y_3[n] = \begin{cases} x[-n/2], & \text{if } n/2 \text{ is an integer} \\ 0, & \text{otherwise} \end{cases}$ $\boxed{\text{odd}}$

12

i) $y[-n] = x[(-n)^2] = x[n^2]$

$y[n] = x[n^2]$

> since $y[-n] = y[n]$, $y[n]$ is even

ex: $y[-2] = x[4] = y[2]$

ii) $y_2[n] = x[3n] = -x[-3n]$

$y_2[-n] = x[-3n] =$

> since $y[n] = -y[-n]$, y is odd

iii) $y_3[n] = \begin{cases} x[-n/2] & n \text{ is even} \\ 0 & \text{else} \end{cases} \rightarrow -\begin{cases} x[n/2] & n \text{ is even} \\ 0 & \text{else} \end{cases} = -y_3[-n]$

$y_3[-n] = \begin{cases} x[n/2] & n \text{ is even} \\ 0 & \text{else} \end{cases}$

since $y_3[n] = -y_3[-n]$

y_3 is odd

(b) **(20 points)** Determine whether the system

$$y[n] = |x[n]|$$

where $|x[n]|$ is the magnitude of the input $x[n]$, is

(i) Linear NOT Linear

(ii) Causal Causal

(iii) Stable Yes

(iv) Time invariant Yes

20

$\alpha x_1 + \beta x_2 \to y_3$

i) $y_3[n] = |\alpha x_1[n] + \beta x_2[n]| \neq \left(\alpha |x_1[n]| + \beta |x_2[n]| = \alpha y_1[n] + \beta y_2[n]\right)$

α could be negative, $|a+b| \neq |a| + |b|$

NOT Linear

ii) Causal, $y[n]$ only depends on current values of $x[n]$

iii) Stable, if $|x[n]| < \infty = B_x$, $y = |B_x| \to y$ is also bounded

iv) Delay input $\to$ $|x[n-m]| = y[n-m]$

Delay output $y[n-m] = |x[n-m]|$

yeilds same result

Yes, TI since delaying input or output yeilds same result upsample, lose data

(c) **(8 points)** Assume that we are given the values of $x[2n]$ and $x[n^2]$ for all n: can we reconstruct the signal $x[n]$ using these values?

With $x[2n]$ we can only recover $x[n]$ for even values of n.

With $x[n^2]$ we can only recover $x[n]$ for $\pm n$ such that n was a perfect square.

We are unable to recover $x[n]$ for values that are not even or perfect squares.

ie, $x[3]$ cannot be recovered.

FALSE

8

Problem 2 (35 points) : The following two questions are not related.

(a) **(20 points)** The response of an LTI system to the input $x_1[n] = u[n]$ is $y_1[n] = a^n u[n]$ where $a < 1$. What is the response of the system when the input is $x_2[n] = b^n u[n]$ where $b < 1$?

You may find the formula for finite geometric sum useful: $\sum_{j=p}^{q} \beta^j = \frac{\beta^p - \beta^{q+1}}{1-\beta}$.

$x_1[n] = u[n]$
$x_1[n-1] = u[n-1]$ $\rightarrow$ $x_1[n] - x_1[n-1] = \delta[n] \rightarrow a^n u[n] - a^{n-1} u[n-1]$

$h[n] = a^n [u[n] - a^{-1} u[n-1]]$

$x_2[n] = b^n u[n] \rightarrow y_2[n]$

$y_2[n] = h[n] * x_2[n]$

$= \sum_{k=-\infty}^{\infty} a^k [u[k] - a^{-1} u[n-1]] b^{n-k} u[n-k]$

$= \sum_{k=-\infty}^{\infty} u[k] u[n-k] a^k b^{n-k} - a^{-1} \sum_{k=-\infty}^{\infty} a^k u[k-1] u[n-k] b^{n-k}$

$= \begin{cases} b^n \sum_{k=0}^{n} a^k b^{-k} & 0 \le k \le n \\ 0 & \text{else} \end{cases} - \begin{cases} b^n a^{-1} \sum_{k=1}^{n} a^k b^{-k} & 1 \le k \le n \\ 0 & \text{else} \end{cases}$

$= \begin{cases} b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k & n \ge 0 \\ 0 & \text{else} \end{cases} - \begin{cases} b^n a^{-1} \sum_{k=1}^{n} \left(\frac{a}{b}\right)^k & n \ge 1 \\ 0 & \text{else} \end{cases}$

20

$= b^n \left(\frac{\left(\frac{a}{b}\right)^0 - \left(\frac{a}{b}\right)^{n+1}}{1 - \left(\frac{a}{b}\right)} \right) u[n] - \frac{b^n}{a} u[n-1] \left(\frac{\left(\frac{a}{b}\right) - \left(\frac{a}{b}\right)^{n+1}}{1 - \left(\frac{a}{b}\right)} \right)$

$$\boxed{= \frac{b^n}{1 - \left(\frac{a}{b}\right)} \left(1 - \left(\frac{a}{b}\right)^{n+1}\right) u[n] - \frac{b^n}{a\left(1 - \frac{a}{b}\right)} \left(\frac{a}{b} - \left(\frac{a}{b}\right)^{n+1} \right) u[n-1]}$$

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You may find the formula for finite geometric sum useful: $\sum_{j=p}^{q} \beta^j = \frac{\beta^p - \beta^{q+1}}{1-\beta}$.

$$x_1[n] = u[n] \quad > \quad x_1[n] - x_1[n-1] = \delta[n] \rightarrow a^n u[n] - a^{n-1} u[n-1]$$

$$x_1[n-1] = u[n-1] \qquad h[n] = a^n \left[u[n] - a^{-1} u[n-1] \right]$$

$$x_2[n] = b^n u[n] \rightarrow y_2[n]$$

$$y_2[n] = h[n] * x_2[n]$$

$$= \sum_{k=-\infty}^{\infty} a^k \left[u[k] - a^{-1} u[n-1] \right] b^{n-k} u[n-k]$$

$$= \sum_{k=-\infty}^{\infty} u[k] u[n-k] a^k b^{n-k} - a^{-1} \sum_{k=-\infty}^{\infty} a^k u[k-1] u[n-k] b^{n-k}$$

$$= \begin{cases} b^n \sum_{k=0}^{n} a^k b^{-k} & 0 \le k \le n \\ 0 & \text{else} \end{cases} - \begin{cases} b^n a^{-1} \sum_{k=1}^{n} a^k b^{-k} & 1 \le k \le n \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k & n \ge 0 \\ 0 & \text{else} \end{cases} - \begin{cases} b^n a^{-1} \sum_{k=1}^{n} \left(\frac{a}{b}\right)^k & n \ge 1 \\ 0 & \text{else} \end{cases}$$

20

$$= b^n \left(\frac{\left(\frac{a}{b}\right)^0 - \left(\frac{a}{b}\right)^{n+1}}{1 - \left(\frac{a}{b}\right)} \right) u[n] - \frac{b^n}{a} u[n-1] \left(\frac{\left(\frac{a}{b}\right) - \left(\frac{a}{b}\right)^{n+1}}{1 - \left(\frac{a}{b}\right)} \right)$$

$$= \boxed{\frac{b^n}{1 - \left(\frac{a}{b}\right)} \left(1 - \left(\frac{a}{b}\right)^{n+1} \right) u[n] - \frac{b^n}{a\left(1 - \frac{a}{b}\right)} \left(\frac{a}{b} - \left(\frac{a}{b}\right)^{n+1} \right) u[n-1]}$$

(b) **(15 points)** Consider two periodic signals $x[n]$ and $y[n]$, both with period N. Define the signal $z[n]$ as the periodic convolution of $x[n]$ and $y[n]$, i.e.,

$$z[n] = x[n] \circledast y[n]$$

Given the signal $z[n]$ and the DTFS spectrum of $y[n]$, is it always possible to <u>uniquely</u> determine the signal $x[n]$? Explain why/why not?

let $z(n)$ have DTFS coefficents ℓ_k and $y(n)$ have d_k.
and $x(n) \rightarrow c_k$

$\ell_K = N c_K d_k$

NO, consider this case where $x(n)$ has fundamental period N and $y(n)$ has fundamental period $N_1 < N$ but there exists K such that $KN_1 = N$. Using the formula above, d_k may have some values k_1 such that $d_{k_1} = 0$ but $c_{k_1} \neq 0$. This creates an issue where it is impossible to determine uniquely what c_k was. If c_k is NOT unique, then there must be multiple ways to determine $x(n)$ using the synthesis equation.

15

Problem 3 (25 points) : Let us say there is a *real* periodic signal $x[n]$, with period $N = 7$. You do not know the signal and wish to determine it. However, you happen to know that the average value of the signal over each period is 0, i.e.,

$$\frac{1}{7}\sum_{n=0}^{6} x[n] = 0$$

Assume that the DTFS coefficients of $x[n]$ are $c_k,\ k \in \{0, 1, ..., 6\}$. Of course, you do NOT know these coefficients. As you are trying to determine the signal, your friend tells you that he/she tried to reconstruct $x[n]$ using its first four DTFS coefficients, i.e., your friend's reconstruction is

$$\tilde{x}[n] = \sum_{k=0}^{3} c_k e^{j\frac{2\pi}{7}kn}$$

Your friend agreed to give you $\tilde{x}[n]$, but not any of the coefficients c_k. After thinking for a few minutes, you realize that $x[n]$ can be obtained using your friend's reconstruction $\tilde{x}[n]$. Explain and derive the expression of $x[n]$ in terms of $\tilde{x}[n]$?

$$\sum_{n=0}^{6} x[n] = 0 \quad \rightarrow \quad c_k = \frac{1}{7}\sum_{n=0}^{6} x[n]\, e^{j(\frac{2\pi}{7}kn)}$$

$$= \frac{1}{7}\left[x[0] + x[1]\, e^{j(\frac{2\pi k}{7})} + \dots x[6]\, e^{j(\frac{2\pi 6k}{7})} \right]$$

$c_k \rightarrow c_0 = 0$ since all complex exponentials are 1

$\tilde{x}[n]$ gives first 4 c_k

$$x[n] = \sum_{k=0}^{6} c_k e^{j\frac{2\pi}{7}kn}$$

$c_{N+k} = c_k$

$c_{-1} = c_{7-1} = c_6$

$c_{-3} = c_{7-3} = c_4$

$$x[n] = \sum_{k=0}^{3} c_k e^{j(\frac{2\pi}{7})kn} + \sum_{k=-3}^{-1} c_k e^{j(\frac{2\pi}{7})kn}$$

since $c_0 = 0$

let $l = -k$

$$= \sum_{k=1}^{3} c_k e^{j(\frac{2\pi}{7})kn} + \sum_{l=1}^{3} c_{-k} e^{j(\frac{2\pi}{7})-ln}$$

since x is real $c_k^* = c_{-k}$

$$= \tilde{x}[n] + \sum_{l=1}^{3} c_l^* e^{-j(\frac{2\pi}{7})(l)(n)}$$

$$= \tilde{x}[n] + \sum_{l=1}^{3} c_k^* \varphi_k^* = \boxed{\tilde{x}[n] + \tilde{x}^*[n]}$$

25