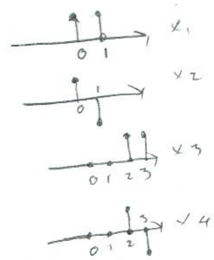
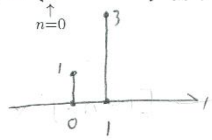
Problem 1 (24 points): Consider the four signals:

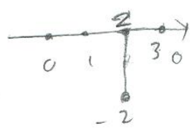


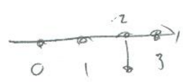
$$x_1[n] = \delta[n] + \delta[n-1]$$
$$x_2[n] = \delta[n] - \delta[n-1]$$
$$x_3[n] = \delta[n-2] + \delta[n-3]$$
$$x_4[n] = \delta[n-2] - \delta[n-3]$$

(a) **(8 points)** Express the signal $x[n] = \{\underset{\substack{\uparrow \\ n=0}}{1}, 3, -1, 0\}$ as a function of the signals $x_i[n]$.

$x_1[n] - x_2[n] + x_1[n] \longrightarrow$  ✓

$\Rightarrow 2x_1[n] - x_2[n]$

$-x_3[n] - x_4[n] \longrightarrow$ 

$\rightarrow -\frac{1}{2}(x_3[n] + x_4[n]) \longrightarrow$  ✓

Thus $\boxed{y[n] = 2x_1[n] - x_2[n] - \frac{1}{2}(x_3[n] + x_4[n])}$

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$y[n] = \{1, 3, -1, 0\}$

(b) **(8 points)** Consider again the signals $x_i[n]$ $i = 1, 2, 3, 4$ and $x[n]$ from part (a) of this problem. You are given that the convolution of the signals $x_1[n]$ with $y[n]$ gives the signal $z_1[n]$, that is,

$$z_1[n] = x_1[n] * y[n] \quad \delta[n] * y[n] + \delta[n-1] * y[n]$$

$y[n] + y[n-1]$

You are also given that

$$z_2[n] = x_4[n] * y[n] \quad \delta[n-2] * y[n] - \delta[n-3] * y[n]$$

$y[n-2] - y[n-3]$

Calculate the convolution $x[n] * y[n]$ as a function of $z_1[n]$ and $z_2[n]$.

$$x[n] * y[n] = \left(2x_1 - x_2 - \tfrac{1}{2}(x_3 + x_4)\right) * y[n]$$

$$= \left(2(\delta[n] + \delta[n-1]) - (\delta[n] - \delta[n-1]) - \tfrac{1}{2}(2\delta[n-2])\right) * y[n]$$

$$= (2\delta[n] + 2\delta[n-1] - \delta[n] + \delta[n-1] - \delta[n-2]) * y[n]$$

$$= (\delta[n] + 3\delta[n-1] - \delta[n-2]) * y[n]$$

$$= y[n] + 3y[n-1] - y[n-2]$$

$$= z_1[n](1 + 3\delta[n-1]) + z_2[n](-\delta[n-2])$$

$z_1[n] = y[n] + y[n-1]$

$z_2[n] = y[n-2] - y[n-3]$

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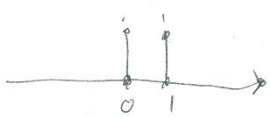
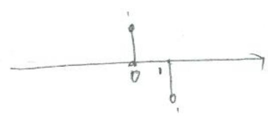
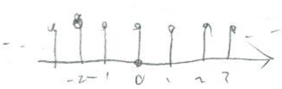
(c) **(8 points)** The periodic extension of a signal $x[n]$ that has length N, is defined to be the signal $x_p[n]$ that simply repeates $x[n]$ every N samples, that is,

$$x_p[n] = \sum_{k=-\infty}^{\infty} x[n-kN]$$

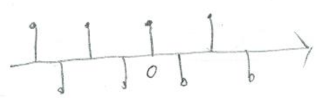
(i) **(4 points)** Consider the periodic extension of the signals $x_1[n]$ and $x_2[n]$ that are given in part (a) of this problem. Are any of these signals even for $N = 2$? If yes which ones?

a. $x_1[n]$  $x_2[n]$ 

$x_{1p}[n] \longrightarrow$ 

$x_{1p}[-n] = x_{1p}[n] \rightarrow$ even for all $n \neq 0$

but @ $n=0$, $x_{1p}[-0] \neq -x_{1p}[0] \neq 0$ not even

$x_{2p}[n] \longrightarrow$ 

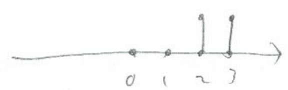
$x_{2p}[0] \neq -x_{2p}[-0]$, not even

$\boxed{\text{neither even}}$ 0

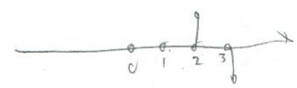
(ii) **(4 points)** Now consider the periodic extension of the signals $x_3[n]$ and $x_4[n]$ that are given in part (a) of this problem. Are any of these signals even for $N = 4$? If yes which ones?

4

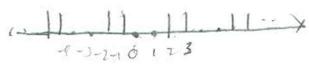
a, x_3



x_4

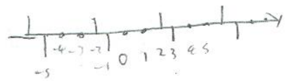


$x_{3,p} \rightarrow$



$x[1] \neq x[-1]$, not even

$x_{4,p} \rightarrow$



$x[1] \neq x[-1]$, not even

Neither even

Problem 2 (40 points) : The following questions are not related to each other.

(a) **(8 points)** A one-dimensional linear classifier takes as input a feature value x and outputs 1 if x is larger than a constant A and zero otherwise.

(i) **(4 points)** Is this system linear?

$y = 1$ for $x > A$, $\rightarrow y[n] = u[x-A]$ where $u[0]=0$

$y[n] = u[x[n]-A]$

Let $x_1 = \left(\frac{2}{3}A\right)$ and $x_2 = \left(\frac{2}{3}A\right)$

such that $y_1 = 0$, $y_2 = 0$

$\rightarrow$ Let $x_3 = \alpha x_1 + \beta x_2 = \frac{2}{3}\alpha A + \frac{2}{3}\beta A$

where α and β are positive integers

$\rightarrow$ Then $y_3 = u\left(\frac{2}{3}A(\alpha+\beta)\right) \rightarrow = 1$ since $\alpha + \beta \geq 2 \rightarrow$

$\neq \alpha y_1 + \beta y_2 = 0$

Not Linear

4

(ii) **(4 points)** Is the system time invariant?

Y

Let $x_k = x[n-k]$

so $y_k = u[x_k - A]$ where $u[0]=0$

$\rightarrow y = u[x[n-k]-A] = y[n-k]$

Thus, the system is $\boxed{\text{Time Invariant}}$

4

(b) **(12 points)** Assume that the system $y[n] = f(x[n])$, where $f()$ is some unknown function, is Linear Time Invariant (LTI). Can you determine if the following systems are LTI? Briefly explain why.

(i) **(4 points)** $z[n] = y[n-5] + y[n-2]$

Linearity: Let $x_1 = a x[n] \xrightarrow{f()} y[n]$

$x_2 = b x[n] \xrightarrow{f()} y[n]$

$x_3[n] = x_1 + x_2 \xrightarrow{LTI} y_3[n] = a y_1 + b y_2$

$y_3[n] \longrightarrow z_3[n] = y_3[n-5] + y[n-2]$

$= a y_1[n-5] + b y_2[n-5] = a z_1 + b z_2$

where $a y_1[n] \rightarrow a z_1[n]$, $a y_2[n] \rightarrow a z_2[n]$

Linear

Time Variance: Let $x_k = x[n-k]$

$\xrightarrow{f()} y_k \overset{LTI}{=} y[n-k]$

$y[n-k] \longrightarrow y[n-k-5] + y[n-k-2]$

$= z[n-k]$ ✓ TI

The sum of two LTI system outputs in itself is also $\boxed{LTI}$.

4

(ii) **(4 points)** $z[n] = y^2[n]$ $= \left[f(x)\right]^2$

Linearity: Let $y_1 = a\,y[n] \rightarrow z_1[n] = a^2 y^2[n]$

$y_2 = b \cdot y[n] \rightarrow z_2[n] = b^2 y^2[n]$

As can be observed, $z_1[n] \neq a\,z[n]$

$z_2[n] \neq b\,z[n]$

$\underline{a} \cdot x[n] \rightarrow a y[n] \rightarrow \underline{a^2} z[n]$

Not linear

Not Linear

Thus, $\boxed{\text{not LTI}}$

4

(iii) **(4 points)** $z[n] = y[2n]$

Linearity: Let $y_1 = a y[n]$

Let $y_2 = b y[n]$

$\rightarrow y_3 = y_1 + y_2 \longrightarrow z_3 = a y[2n] + b y[3n]$

$= a z_1[n] + b z_2[n]$

Linear ✓ Thus, $a x_1 + b x_2 \xrightarrow{f()} a y_1 + b y_2 \rightarrow a z_1 + b z_2$

Time Variance: Let $y_k = y[n-k]$

$y_k \rightarrow z_k = y_k[2n] = y[2n-k] \neq z[n-k] = y[2n-2k$

Thus $x[n-k] \xrightarrow{f()} y[n-k] \not\rightarrow z[n-k]$

Not TI,

Therefore $\boxed{\text{Not LTI}}$

4

$w[n] \rightarrow \boxed{D} \rightarrow w[n-1]$

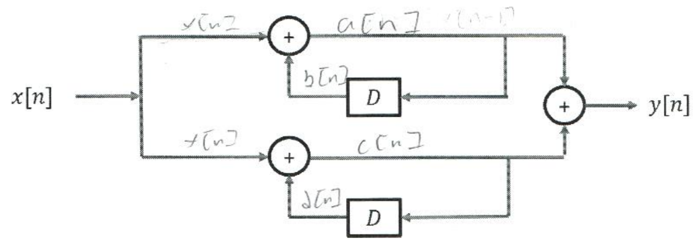


Figure 1: System for Problem 2(c)

(c) **(20 points)** Consider a relaxed system as shown in Fig. 1

(i) **(8 points)** Can you write the input-output equations for this system?

$$y[n] = a[n] + c[n]$$

$$= x[n] + b[n] + x[n] + d[n]$$

$$= 2x[n] + a[n-1] + c[n-1]$$

$$= 2x[n] + (x[n-1] + b[n-1]) + (x[n-1] + d[n-1])$$

$$= 2x[n] + 2x[n-1] + a[n-2] + c[n-2]$$

- - - -

$$= 2x[n] + 2x[n-1] + 2x[n-2] - - -$$

$$\boxed{y[n] = 2\sum_{k=0}^{\infty} x[n-k]}$$

3+5

(ii) **(4 points)** Is this a BIBO stable system? (explain why?)

if $|x[n]| \leq B$ for all n,

then $y[n] = 2\sum_{k=0}^{\infty} x[n-k]$

↳ Maximum $\rightarrow 2\sum_{k=0}^{\infty} B = \infty \cdot B = \infty$

$\boxed{\text{Not BIBO stable}}$

4

(iii) **(8 points)** Assume you connect two of these relaxed systems, shown in Fig. 1, in series, what is the impulse response of the overall equivalent system?

$x[n] \longrightarrow y[n] \longrightarrow z[n]$

$= 2\sum_{k=0}^{\infty} x[n-k]$ $\qquad = 2\sum_{j=0}^{\infty} y[n-j]$

$z[n] = \sum_{j=0}^{\infty} y[n-j] = 2\sum_{j=0}^{\infty}\left(2\sum_{k=0}^{\infty} x[n-j-k]\right) = 4\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} x[n-j-k]$

$h_1[n] = 2\sum_{k=0}^{\infty} \delta[n-k] = 2u[n]$

$h_2[n] = 2\sum_{k=0}^{\infty} 2u[n-k] = 4\sum_{k=0}^{\infty} u[n-k]$

$= \boxed{4r[n]} = h_2[n]$

2

Problem 3 (36 points) : The following questions are not related to each other.

(a) **(14 points)** True or false: Let $x[n]$ be a real periodic signal with even period N, and c_k be the coefficients of its associated discrete-time Fourier series. Then, there always exist j_1, j_2 such that c_{j_1} and c_{j_2} are purely real. Mathematically justify your answer.

a. $C_k = \langle x[n], \phi_k[n] \rangle$

$\phi_{j_1} = e^{j\frac{2\pi}{N}j_1 n}$, $\phi_{j_2} = e^{j\frac{2\pi}{N}j_2 n}$

$\frac{2\pi}{N} j_1 = \pi$

$\to 2\pi j_1 = N\pi$

$j_1 = \frac{N}{2}$

for ϕ_{j_1}, ϕ_{j_2} to be real,

then $\frac{2\pi}{N}j_1 = k_1\pi$, $\frac{2\pi}{N}j_2 = k_2\pi$ where k_1, k_2 are integers

$\to 2\pi j_1 = N k_1 \pi$ $\quad \to 2\pi j_2 = N k_2 \pi$

$\to j_1 = \frac{N k_1}{2}$ $\quad \to j_2 = \frac{N k_2}{2}$

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since k_1, k_2 are integers
and N is even,

there will always exist ϕ_{j_1}, ϕ_{j_2} which are purely real,

and b/c $c_k = \frac{1}{N}\sum_{k=0}^{N-1} x[n]\phi_k[n]$ where $x[n]$ is real, there will always exist c_{j_1}, c_{j_2} which are purely real!

You also need to find j_1 & j_2 that satisfies this.

$\frac{2\pi}{2\pi/3} = 3$ $\frac{2\pi}{\frac{\pi}{3}} = 3/2$

$T=3$

(b) **(8 points)** Find the discrete-time Fourier Series coefficients for the following signal:

$$x[n] = \sin(\frac{2\pi n}{3})\cos(\frac{\pi n}{3})$$

Period:

$\sin(\frac{2\pi}{3}n)\cos(\frac{\pi n}{3})$

$= \frac{1}{4j}\left(e^{j\frac{2\pi}{3}n} - e^{-j\frac{2\pi}{3}n}\right)\left(e^{j\frac{\pi}{3}n} + e^{-j\frac{\pi}{3}n}\right)$

$= \frac{1}{4j}\left(e^{j\pi n} + e^{j\frac{\pi}{3}n} - e^{-j\frac{\pi}{3}n} + e^{-j\pi n}\right)$

$C_0 = 0$

$C_1 = \frac{1}{4j}$

$C_{-1} = -\frac{1}{4j}$

$C_4 = \frac{1}{4j}$

$C_{-4} = -\frac{1}{4j}$

6

(c) **(14 points)** Assume that for a periodic signal $x[n]$ with ~~period N~~ you find the DTFS coefficients c_k. As we discussed in class, the Fourier Series coefficients c_k can also be thought as the values of a periodic signal with the same period N. Lets call this periodic signal $c[k]$, that is,

$$c[k] = c_k$$

Your friend claims that if you take the Fourier Series expansion of this periodic signal $c[k]$, you will get coefficients d_m that will be sufficient to retrieve the original signal $x[m]$. Are they right? Mathematically justify your answer.

$$c[k] = \frac{1}{N} \sum_{k=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

$$d_m = \frac{1}{N} \sum_{k=0}^{N-1} c[k]\, e^{-j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left(\frac{1}{N} \sum_{m=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}mn} \right) e^{-j\frac{2\pi}{N}kn}$$

8

$$d_m[n] = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}n(m+k)}$$

$$c[k] = \sum_{m=0}^{N-1} d_m\, e^{-j\frac{2\pi}{N}mn}$$

$$x[n] = \sum_{k=0}^{N-1} c_k\, e^{-j\frac{2\pi}{N}}$$