Total: 100 points

ECE113: Digital Signal Processing

Midterm 1

12:00 pm - 1:40 pm, Apr 24, 2019

NAME: ______________________________ **UID**: ________________

This exam has 3 problems, for a total of 100 points.

Closed book. No calculators. No electronic devices.
One page, letter-size, one-side cheat-sheet allowed.
Answer the questions in the space provided below each problem. If you run out of room for an answer, continue on the back of the page or use the extra pages at the end.
Please, write your name and UID on the top of each loose sheet!
GOOD LUCK!

Problem	Points	Total Points
1		24
2		40
3		36
Total		100

Extra Pages: ________
To fill in, in case extra sheets are used apart from what is provided.

Note: Answers without justification will not be awarded any marks.

Problem 1 (20 points): Consider the four signals:

$$x_1[n] = \delta(n) + \delta(n-1)$$
$$x_2[n] = \delta(n) - \delta(n-1)$$
$$x_3[n] = \delta(n-2) + \delta(n-3)$$
$$x_4[n] = \delta(n-2) - \delta(n-3)$$

(a) **(8 points)** Express the signal $x[n] = \{1, 3, -1, 0\}$ as a function of the signals $x_i[n]$.
Solution: We can rewrite $x[n] = \{1, 3, -1, 0\}$ as $x[n] = \delta(n) + 3\delta(n-1) - \delta(n-2) + 0\delta(n-3)$, and the functions $\delta(i)$ can be generated by the signals x_i as follows:

$$\delta(n) = \frac{1}{2}(x_1[n] + x_2[n])$$
$$\delta(n-1) = \frac{1}{2}(x_1[n] - x_2[n])$$
$$\delta(n-2) = \frac{1}{2}(x_3[n] + x_4[n])$$
$$\delta(n-3) = \frac{1}{2}(x_3[n] - x_4[n])$$

Hence $x[n] = \frac{1}{2}(x_1[n] + x_2[n]) + \frac{3}{2}(x_1[n] - x_2[n]) - \frac{1}{2}(x_3[n] + x_4[n]) = 2x_1[n] - x_2[n] - \frac{1}{2}x_3[n] - \frac{1}{2}x_4[n]$.

Common Mistakes:

- Students tried to "guess" the function by empirically trying candidates.
- Many students with wrong answers did not verify the result given by their candidate signal and the stated signal.
- Students needed to fully justify their answer.

(b) **(8 points)** You are given that the convolution of $x_1[n]$ with a signal $y[n]$ equals $z_1[n]$, that is,

$$z_1[n] = x_1[n] * y[n]$$

You are also given that

$$z_2[n] = x_4[n] * y[n]$$

Calculate the convolution $x[n] * y[n]$ as a function of $z_1[n]$ and $z_2[n]$.

Solution: From the previous question, we have

$$x[n] * y[n] = (2x_1[n] - x_2[n] - \frac{1}{2}x_3[n] - \frac{1}{2}x_4[n]) * y[n],$$

which, by distributivity of convolution, also gives

$$x[n] * y[n] = 2x_1[n] * y[n] - x_2[n] * y[n] - \frac{1}{2}x_3[n] * y[n] - \frac{1}{2}x_4[n] * y[n].$$

We also notice that signals $x_2[n]$ and $x_3[n]$ are obtained by shifting time in signals $x_1[n]$ and $x_4[n]$ as follows: $x_3[n] = x_1[n-2]$ and $x_2[n] = x_4[n+2]$. Thus, $x_2[n] * y[n] = x_4[n+2] * y[n] = z_2[n+2]$ and $x_3[n] * y[n] = x_1[n-2] * y[n] = z_1[n-2]$. We can then express the convolution $x[n] * y[n]$ only in terms of $z_1[n]$ and $z_2[n]$:

$$x[n] * y[n] = 2z_1[n] - z_2[n+2] - \frac{1}{2}z_1[n-2] - \frac{1}{2}z_2[n]$$

Common Mistakes:

- A mistake in previous question was not penalizing too much the grade for this question.
- Students were expected to use and emphasize the use of the distributivity property.
- Students needed to explain how to express $x_2[n] * y[n]$ and $x_3[n] * y[n]$ by using $z_1[n]$ and $z_2[n]$ thanks to the introduction of time shift, and how they were getting these specific time shifts. Many students did not manage to find these expressions.
- Some results were using other terms than $z_1[n]$ and $z_2[n]$, which was not what was asked in the statement of the question.

(c) **(8 points)** The periodic extension of a signal $x[n]$ that has length N, is defined to be the signal $x_p[n]$ that simply repeates $x[n]$ every N samples, that is,

$$x_p[n] = \sum_{k=-\infty}^{\infty} x[n - kN]$$

(i) **(4 points)** Consider the periodic extension of the signals $x_1[n]$ and $x_2[n]$ that are given in part (a) of this problem. Are any of these signals even for $N = 2$? If yes which ones?

Solution: Both signals are even. One can see that through a graphical representation, or by giving the expression of the extended signals. We will have $\tilde{x}_1[n] = 1$ and $\tilde{x}_2[n] = (-1)^n$, which are indeed two even signals.

(ii) **(4 points)** Now consider the periodic extension of the signals $x_3[n]$ and $x_4[n]$ that are given in part (a) of this problem. Are any of these signals even for $N = 4$? If yes which ones?

(iii) **Solution:** Similarly as the previous question, a graphical representation shows that none of these signals are even. One can also explicit the closed form expression of the signal and see they are not even functions of n.

Common Mistakes:

- Most students made a graph of the extended signals to visualize them and justify their answer. The most common mistake was a wrong drawing of this graph.

Problem 2 (40 points) : The following questions are not related to each other.

(a) **(8 points)** A one-dimensional linear classifier takes as input a feature value x and outputs 1 if x is larger than a constant A and zero otherwise.

(i) **(4 points)** Is this system linear?

Solution: No, Consider the following example:

$$x_1[n] = 2A\delta[n] \to y_1[n] = \delta[n]$$
$$x_2[n] = 3A\delta[n] \to y_2[n] = \delta[n]$$
$$x_3[n] = 0.1x_1[n] + 0.1x_2[n] = 0.5A\delta[n] \to y_3[n] = 0 \neq 0.1y_1[n] + 0.1y_2[n]$$

(ii) **(4 points)** Is it time invariant?

Solution: Yes. Since the system defines the output based on the input at a particular time instance n, the time-shift in input will also result in the time-shift in output.

$$x[n] \to y[n] \implies x[n-K] \to y[n-K]$$

(b) **(12 points)** Assume that the system $y[n] = f(x[n])$, where $f()$ is some unknown function, is Linear Time Invariant (LTI). Can you determine if the following systems are LTI? Briefly explain why.

(i) **(4 points)** $z[n] = y[n-5] + y[n-2]$

Solution: It is LTI, since we can remark that the system defined through the difference equation above is LTI and it is cascaded with another LTI system, thus making the overall system LTI as well. We can also prove it using the definitions of linearity and time invariance. The overall system satisfies the equation

$$z[n] = f(x[n-5]) + f(x[n-2]) = g(x[n]).$$

Let us show that $g()$ is linear and gives a time invariant system. If we consider two signals $x_1[n]$ and $x_2[n]$, and take any linear combination of them, we have:

$$\begin{aligned} g(\alpha x_1[n] + \beta x_2[n]) &= f(\alpha x_1[n-5] + \beta x_2[n-5]) + f(\alpha x_1[n-2] + \beta x_2[n-2]) \\ &= \alpha f(x_1[n-5]) + \alpha f(x_1[n-2]) + \beta f(x_2[n-5]) + \beta f(x_2[n-2]). \end{aligned}$$

We also have that

$$\alpha f(x_1[n-5]) + \alpha f(x_1[n-2]) + \beta f(x_2[n-5]) + \beta f(x_2[n-2]) = \alpha g(x_1[n]) + \beta g(x_2[n]),$$

which shows the linearity of the overall system.

We verify its time invariance in the same manner: for any $K > 0$, we know that we have $f(x[n-K]) = y[n-K]$; hence, $f(x[n-5-K]) + f(x[n-2-K]) = g(x[n-K])$, which shows the time invariance of the system.

(ii) **(4 points)** $z[n] = y^2[n]$

Solution: The overall system is not LTI since the system above is not Linear. Let us see a counter-example by taking $f()$ as the identity function. Then, $f()$ is indeed linear, and $z[n] = x^2[n]$, which is clearly nonlinear.

(iii) **(4 points)** $z[n] = y[2n]$

Solution: The overall system is not LTI since the system above is not time-invariant. Again, we can have an easy counter-example by taking $f()$ as the identity function: the overall system becomes $z[n] = x[2n]$, which is well-known to not be time-invariant.

Common Mistakes:

- Incorrect application of the definition of linearity, resulting in students proving a nonlinear system was linear.
- Some students forgot to also prove the time invariance after the linearity in order to prove that a system is (or is not) LTI.

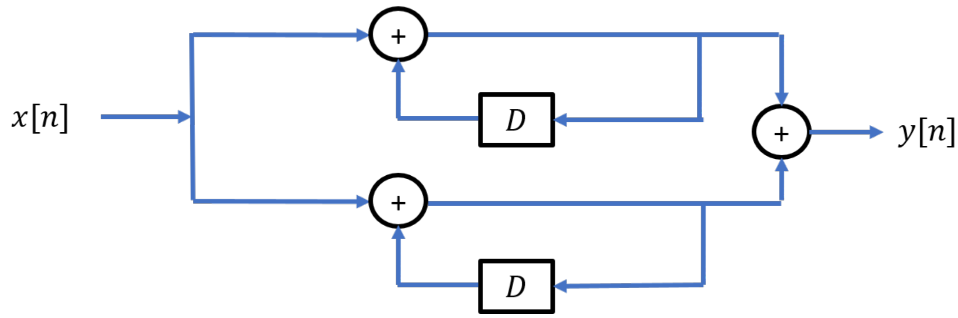

Figure 1: System for Problem 2(c)

(c) **(20 points)** Consider the relaxed system shown in Fig. 1

(i) **(8 points)** Can you write the input-output equations for this system?
Solution: This block diagram is composed of two identical branches that are summed to obtain $y[n]$. Let us denote by $w[n]$ the signal produced by each of these branches. We have $y[n] = 2w[n]$. We also have that $w[n] = x[n] + w[n-1]$, thus the input-output equation associated to the diagram:

$$y[n] = 2x[n] + y[n-1]$$

Common Mistakes:

- Students needed to give a complete justification to have full credit.
- Some students did not apply correctly the effect of the delay block to a signal and introduced a delay D, or considered D as a scalar that multiplies the initial signal.
- Several students did not notice the diagram was made of two identical branches with same intermediate signal, and gave an equation stating two intermediate signals w and z.
- Several students did not introduce an intermediate signal and directly applied a delay to signal $x[n]$.
- Several students gave a non final expression which contained intermediate signal(s).

(ii) **(4 points)** Is this a BIBO stable system? (explain why?)

Solution: The system is not BIBO stable. A simple counter example would be $x[n] = u[n]$, for which the response of the system is $y[n] = 2(n+1)u[n]$.
Here, $|x[n]| \leq 1$ for all n, but $|y[n]|$ is not bounded.

Common Mistakes:

- Wrong statement of the definition of BIBO stable.
- Several students wrongly stated that feedback loops were making the system stable.

(iii) **(8 points)** Assume you connect two of these relaxed systems, shown in Fig. 1, in series, what is the overall equivalent impulse response?

Solution: The impulse response of the equivalent system shown in Fig. 1 is $2u[n]$.

Thus the impulse response of two such relaxed systems connected in series will be

$$2u[n] * 2u[n] = 4(n+1)u[n]$$

Common Mistakes:

- Several students tried to compute the input-output equation of the new system.
- A majority of students did not think of using convolution product for systems in series.
- A majority of students confused the output given by the system to signal $x[n]$ and the impulse response of the system.

Problem 3 (36 points) : The following questions are not related to each other.

(a) **(14 points)** True or false: Let $x[n]$ be a real periodic signal with even period N, and c_k be the coefficients of its associated discrete-time Fourier series. Then, there always exist j_1, j_2 such that c_{j_1} and c_{j_2} are purely real. Mathematically justify your answer.

Solution: We can use the properties of the DTFS coefficients:
We know,

$$c_{-k} = c_{N-k}$$

Also, since $x[n]$ is given to be real, we have

$$c_{-k} = c_k^*$$

Now, lets say that for some k, c_k is real. Therefore, $c_k^* = c_k$.
Using all these results, we have

$$c_{-k} = c_k^* = c_{N-k} = c_k \quad \text{for some k}$$

Thus, $c_{N-k} = c_k$ holds only for $k = 0, \frac{N}{2}$. Since, N is given to be even, the statement is True.

Common Mistakes:

- The period N is given to be even and not the signal $x[n]$
- Both the coefficients j_1 and j_2 should be from the same period. We know that the DTFS coefficients c_k are periodic with period N. So if c_0 is real, clearly, all the other coefficients that are repeated after the period N will be real, i.e., c_{kN} will be real for any integer k.
- If the signal $x[n]$ is real and odd with even period N, then the coefficients c_0 and $c_{\frac{N}{2}}$ will be 0.

(b) **(8 points)** Find the Fourier Series coefficients for the following signal:

$$x[n] = \sin(\frac{2\pi n}{3})\cos(\frac{\pi n}{3})$$

Solution: $x[n]$ is periodic with period 6 (=LCM(3,6)).

$$\begin{aligned} x[n] &= \sin(\frac{2\pi n}{3})\cos(\frac{\pi n}{3}) \\ &= \frac{e^{j2\pi n/3} - e^{-j2\pi n/3}}{2j} \cdot \frac{e^{j\pi n/3} + e^{-j\pi n/3}}{2} \\ &= \frac{e^{j\pi n} + e^{j\pi n/3} - e^{-j\pi n/3} - e^{-j\pi n}}{4j} \\ &= \frac{e^{j2\pi n/6} - e^{-j2\pi n/6}}{4j} \end{aligned}$$

Therefore, we get:

$$c_k = \begin{cases} \frac{1}{4j}, & \text{for } k = 1 \\ \frac{-1}{4j}, & \text{for } k = -1 + 6 = 5 \\ 0, & \text{otherwise} \end{cases}$$

Common Mistakes:

- If $x_1[n]$ is a periodic signal with period N_1, and $x_2[n]$ is a periodic signal with period N_2, then the signal $x[n] = x_1[n]x_2[n]$ is also a periodic signal with a period of $N = \text{LCM}(N_1, N_2)$. Certainly, N_1N_2 is also a period of $x[n]$, but N could be the smallest period.
- $e^{j\pi n} = e^{-j\pi n} = (-1)^n$. Therefore, $c_3 = c_{-3} = 0$.

(c) **(14 points)** Assume that for a periodic signal $x[n]$ with period N you find the DTFS coefficients c_k. As we discussed in class, the Fourier Series coefficients c_k can also be thought as the values of a periodic signal with the same period N. Lets call this periodic signal $c[k]$, that is,

$$c[k] = c_k$$

Your friend claims that if you take the Fourier Series expansion of this periodic signal $c[k]$, you will get coefficients d_m that will be sufficient to retrieve the original signal $x[m]$. Are they right? Mathematically justify your answer.

Solution: The DTFS coefficients c_k for the signal $x[n]$ will be:

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

$$c[k] = c_k \text{ is also periodic with period } N$$

Let d_m be the DTFS coefficients of the signal $c[k]$.

$$\begin{aligned}
d_m &= \frac{1}{N}\sum_{k=0}^{N-1} c[k]e^{-j2\pi km/N} \\
\implies d_{-m} &= \frac{1}{N}\sum_{k=0}^{N-1} c[k]e^{j2\pi km/N} \\
\implies d_{-m} &= \frac{1}{N}x[m] \\
x[m] &= Nd_{-m}
\end{aligned}$$

Common Mistakes:

- $d_{-m} \neq d_m^*$, since we do not know if the coefficients c_k are real or complex.
- In the problem, $x[n]$ is given to be any signal. It is not assumed to be real or even or odd signal.