EE 113 Digital Signal Processing

Spring 2013

Midterm Exam

Closed Book, 1 sheet of notes allowed

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TOTAL _________ / 60

Problem 1 (*10 pts*)
Find the energy and the average power of the following sequence:

$$x(n) = \left(\frac{1}{3}\right)^{n} u(n-1) + \left(\frac{1}{2}\right)^{n-1} u(n-2)$$

Problem 2 (*10 pts*)

What is the sequence that results from sampling $x(t) = \cos(50\pi t)$ at the rate of 150 samples per second? What is the angular frequency and period of the sequence? Can you suggest a different sampling rate of getting the same exact sequence?

Problem 3 (*10 pts*)
A relaxed system is described by the difference equation

$$y(n) - \frac{1}{2} y(n-1) = x^2(n)$$

where $x(n)$ denotes the input sequence and $y(n)$ denotes the output sequences. Prove or give counter-examples:

(a) Is the system linear?
(b) Is the system time-invariant?
(c) Is the system causal?
(d) Is the system BIBO stable?

Problem 4 (*10 pts*)
For the system given below:

$$y(n) = -\frac{1}{4}y(n-1) + x(n-1), \qquad y(-1) = 1, \qquad n \geq 0$$

(a) Find the impulse response sequences of the system.
(b) Sketch the block diagram representation of the system.
(c) Is the system LTI?

Problem 5 (*10 pts*)

The response of a relaxed LTI system to $x(n) = u(n-2)$ is $y(n) = \left(\frac{1}{2}\right)^{n-2} u(n-4)$. Find its impulse response sequence. Is this a BIBO stable system?

Problem 6 (*10 pts*)
For the following system, determine whether or not the system is causal, linear, time-invariant, and memoryless:

$$y(n) = \sum_{k=n-n_o}^{n+n_o} x(k)$$

Problem 1:

$$\mathcal{E}_x = \sum_{n=-\infty}^{+\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{+\infty} \left| \left(\frac{1}{3}\right)^n u(n-1) + \left(\frac{1}{2}\right)^{n-1} u(n-2) \right|^2$$

$$= \sum_{n=-\infty}^{+\infty} \left[\left(\frac{1}{9}\right)^n u(n-1) + \left(\frac{1}{4}\right)^{n-1} u(n-2) + 2\left(\frac{1}{3}\right)^n \left(\frac{1}{2}\right)^{n-1} u(n-2) \right]$$

$$= \sum_{n=1}^{+\infty} \left(\frac{1}{9}\right)^n + 4\sum_{n=2}^{+\infty} \left(\frac{1}{4}\right)^n + 4\cdot\sum_{n=2}^{+\infty} \left(\frac{1}{6}\right)^n$$

$$= \frac{\frac{1}{9}}{1-\frac{1}{9}} + 4\,\frac{\left(\frac{1}{4}\right)^2}{1-\frac{1}{4}} + 4\cdot\frac{\left(\frac{1}{6}\right)^2}{1-\frac{1}{6}}$$

$$= \frac{71}{120} \simeq 0.5917$$

$P_x = 0$ since this sequence is an energy sequence.

2. $\because$ rate = 150 samples / second

$\therefore t = \frac{n}{150}$.

$$x(n) = \cos\left(50\pi \frac{n}{150}\right) = \cos\left(\frac{\pi}{3} n\right)$$

Angular frequency $= \frac{\pi}{3}$ rad/sample

Period $= \frac{2\pi}{\pi/3} = 6$ samples.

Let sampling rate be R samples / second

We want $50\pi \frac{n}{R} = \left(\frac{\pi}{3} + 2k\pi\right) n \quad k \in \mathbb{Z}$

$\Rightarrow R = \frac{150}{1+6k} \quad k \in \mathbb{Z}$

$\therefore R = \frac{150}{7}$ or $\frac{150}{13}$ or $\frac{150}{19}$ or 6 or $\frac{150}{31}$, ... samples/second

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(a). No. Counter-example: $x(n) = \delta(n)$,

Let $y_1(n) = S[x(n)]$. $y_2(n) = S[2x(n)]$

$2y_1(0) = 2 \neq 4 = y_2(0)$

∴ It is not linear.

(b) Yes. Let $y(n) = S[x(n)]$. $y_k(n) = S[x(n-k)]$

$$
\begin{aligned}
y(n) &= x^2(n) + \frac{1}{2} y(n-1) \\
&= x^2(n) + \frac{1}{2} x^2(n-1) + \frac{1}{4} y(n-2) \\
&\vdots \\
&= \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m x^2(n-m)
\end{aligned}
$$

$$
\begin{aligned}
y_k(n) &= x^2(n-k) + \frac{1}{2} y_k(n-1) \\
&= x^2(n-k) + \frac{1}{2} x^2(n-k-1) + \frac{1}{4} y_k(n-2) \\
&\vdots \\
&= \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m x^2(n-k-m) \\
&= y(n-k)
\end{aligned}
$$

∴ It is time-invariant

(c) Yes. ∵ $y(n) = \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m x^2(n-m)$

By definition, it depends only on $x(k)$ for $k \leq n$.

∴ It is causal.

(d) Yes. Let $|x(n)| \le B_x < \infty$. $\forall n$.

$$|y(n)| = \left| \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m x^2(n-m) \right|$$

$$\le \sum_{m=0}^{\infty} \left| \left(\frac{1}{2}\right)^m x^2(n-m) \right|$$

$$\le \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m B_x^2$$

$$= 2 B_x^2 < \infty$$

$\therefore$ It is BIBO stable.

Problem 4.

(a) We use the zero-state + zero-input method to solve for the impulse response of the system

1) The zero-state part:

$$y(n) = -\frac{1}{4}y(n-1) + \delta(n-1), \text{ relaxed}$$

The characteristic function is given by $\lambda = -\frac{1}{4}$

The general solution is:

$$y_h(n) = C\lambda^n = C\cdot\left(-\frac{1}{4}\right)^n, \quad n \geq 2$$

When $n=0$: $y(0) = -\frac{1}{4}\cdot 0 + 0 = 0$

$n=1$: $y(1) = -\frac{1}{4}\cdot 0 + 1 = 1$

$n=2$: $y(2) = -\frac{1}{4}\cdot 1 + 0 = -\frac{1}{4} = C\cdot\left(-\frac{1}{4}\right)^2$

$\Rightarrow C = -4$

Thus,

$$y_{zs}(n) = \left\{\begin{array}{ll} 0, & n \leq 0 \\ 1, & n = 1 \\ \left(-\frac{1}{4}\right)^{n-1}, & n \geq 2 \end{array}\right\} = \left(-\frac{1}{4}\right)^{n-1}u(n-1)$$

2) The zero-input part.

$$y(n) = -\frac{1}{4}y(n-1), \quad y(-1) = 1$$

Using the general solution, we get:

$$y_h(n) = C\lambda^n = C\left(-\frac{1}{4}\right)^n, \quad \forall n$$

When $n=-1$: $C\left(-\frac{1}{4}\right)^{-1} = y(-1) = 1 \Rightarrow C = -\frac{1}{4}$

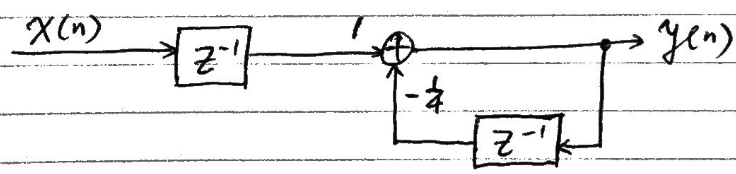
Thus, $y_{zi}(n) = (-\frac{1}{4})^{n+1}$, $\forall n$

3) We conclude that, for $n \geq 0$:

$$y(n) = y_{zs}(n) + y_{zi}(n)$$
$$= (-\frac{1}{4})^{n-1} u(n-1) + (-\frac{1}{4})^{n+1} u(n)$$
$$= -\frac{1}{4}\delta(n) + 17(-\frac{1}{4})^{n+1} u(n-1)$$

where we set $y_{zi}(n) = (-\frac{1}{4})^{n+1} u(n)$ for $n \geq 0$ since this is satisfied by $y_{zi}(n) = (-\frac{1}{4})^{n+1}$, $\forall n$ (and we don't care about the values for $n < 0$ — see the problem statement)

(b)



(c) The system is NOT LTI since the non-zero initial conditions.

5. $S[\delta(n)] = S[u(n) - u(n-1)] = S[u(n)] - S[u(n-1)]$

$$= S[x(n+2)] - S[x(n+1)] = y(n+2) - y(n+1)$$

$$= \left(\frac{1}{2}\right)^n u(n-2) - \left(\frac{1}{2}\right)^{n-1} u(n-3)$$

$$= \frac{1}{4}\delta(n-2) + \left[\left(\frac{1}{2}\right)^n - \left(\frac{1}{2}\right)^{n-1}\right] u(n-3)$$

$$= \frac{1}{4}\delta(n-2) - \left(\frac{1}{2}\right)^n u(n-3)$$

$$\sum_{n=-\infty}^{\infty} \left| \frac{1}{4}\delta(n-2) - \left(\frac{1}{2}\right)^n u(n-3) \right|$$

$$= \frac{1}{4} + \sum_{n=3}^{\infty} \left(\frac{1}{2}\right)^n$$

$$= \frac{1}{4} + \frac{\frac{1}{8}}{1 - \frac{1}{2}}$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2} < \infty$$

It is BIBO stable.

Problem 6

1) if $n_0 \geq 1$, then the system is described by.

$$y(n) = \sum_{k=n-n_0}^{n+n_0} x(k)$$

$$= x(n-1) + x(n) + x(n+1) + \cdots$$

The system is non-causal, linear, time-invariant, and dynamic.

2) if $n_0 = 0$, then $y(n) = x(n)$

The system is causal, linear, time-invariant, and memoryless.